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STOCHASTIC MODELS FOR HETEROGENEOUS OPINION DYNAMICS

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To my family.

ABSTRACT

The aim of this dissertation is to model and study the effect of heterogeneous opinion dynamics on graphs. This work will be mostly committed to analyzing the effect of anti social behaviours in competition with dynamics that would normally push the system towards order and consensus. The analysis will be mainly carried out under the mean field assumption and with voter and majority laws as building blocks. The first models that we are going to deduct and analyze, will be characterized by individuals who behave in different ways independently on who they are interacting with; instead, other models that we will develop later on will take into account this dependence. In the last part of this work, we will investigate an interesting heterogeneous model involving a full majority dynamic, which will bring to interesting considerations about ODE's with discontinuous right-hand side and the validity of the Kurt's theorem. Finally, we will go beyond the mean field by considering a model describing anti social behaviours on a star graph.

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Part I

INTRODUCTION

1

INTRODUCTION TO STOCHASTIC MODELING OF SOCIAL DYNAMICS

1.1 A STOCHASTIC APPROACH TO SOCIAL DYNAMICS

The concept that many laws of nature share a statistical origin is so deeply established in almost all fields of modern physics, that statistical physics has acquired the status of a discipline on its own. Given their success and their very general conceptual framework, in recent years there has been a trend toward applications of stochastic methods to interdisciplinary fields as they played a pivotal role in understanding dynamics in biology, medicine, information technology, computer science etc.. Indeed, there are several examples of phenomena sharing a mathematical description with probabilistic notions at its core, like epidemics diffusion, images spreading on the web, the study of gene frequencies among a population of reproducing individuals (description of genetic models can be found in [1]) or the persistence of particular species in an ecosystem and many others (an extensive review of the state of the art of stochastic models for social dynamics can be found in [2]) .

In particular, a growing interest raised towards applications of stochastic models to describe social phenomena, a field apparently very far from the orginal domains of application of statistical methods. Simple ideas of statistical description of social phenomena are actually rather old, and they anticipated the use of probabilistic and statistic methods to model physic phenomena. One of the very first use of simple statistics method to extract useful information from the collective properties of a large number of individuals was the analysis of birth (and death) rate in certain population, which is an issue of concern and policy for national governments that may seek to increase or reduce it in order to pursuit a certain sustainability strategy. More refined statistical models have been developed later on to describe crime statistics. All these studies suggest that there exist very general laws emerging from the collective behaviour of a number of interacting people and led many scientists and philosophers to call for some quantitative understanding (in the sense of physics) on how such precise regularities arise out of the apparently erratic behavior of single individuals.

In social phenomena, the basic constituents are people rather than particles, and single individuals interact with a certain number of peers, usually negligible compared to the whole population. Despite the presence of these "local interacting dynamics", human societies are characterized by stunning global regularities arising in many different aspects. Some of the most interesting and relevant of these regularities are self-organization behaviours, like transitions from disorder to order. Examples of these spontaneous transitions are the formation of opinion consensus about a specific issue starting from a very disorderd configuation of disagreement, which may lead to the emergence of coalitions among people; the spontaneous rise of a new culture or the creation of a new common language. Even the spread of innovation as well as deseases among a population show very regular patterns.

During the last century, the world has undergone dramatic changes that deeply affected social dynamics. This is mainly due to the huge diffusion of new ways to communicate and interact with each other. In particular, the creation of the web and the subsequent birth of social networks have contributed to create the so called *connected world*, which allows to have interactions not only between people belonging to the same community but even between people spatially very far from each other [3]. All this increased connectivity makes the study of social phenomena much more significant and challenging than in the past, leading to the construction of a systematic framework, based on stochastic models, to face and analyze these kind of problems.

That said, approaching the studying of social dynamics leads immediately to some conceptual difficulties. Similarly to what is usually done in the analysis of physical phenomena, one may decide to study a certain dynamic at different scales. The microscopic approach takes into account the dynamics of each elementary consituent that are coupled according to the corresponding interactions. Conversely, a macroscopic point of view describes the system dynamic in terms of local avarages of microscopic quantities (for instance, the description of a fluid or a traffic flow in terms of particles density). Usually in physics, the microscopic point of view is chosen to describe the dynamic of a single or few elements, while the macroscopic scale is used when many constituents interact with each other. In common applications, the elementary components of the systems investigated, atoms and molecules, are relatively simple objects, whose behavior is very well known: the macroscopic phenomena are not due to a complex behavior of single entities, rather to nontrivial collective effects resulting from the interaction of a large number of "simple" elements.

Humans are exactly the opposite of such simple entities: the detailed behavior of each of them is already the complex outcome of many physiological and psychological processes, still largely unknown. No one knows precisely the dynamics of a single individual, nor the way he interacts with others. Moreover, even if one knew the very nature of such dynamics and such interactions, they would be much more complicated than, say, the forces that atoms exert on each other. It would be impossible to describe them precisely with simple laws and few parameters and a classic microscopic approach is unachievable. Therefore any modeling of social dynamics inevitably involves a huge and unwarranted simplification of the real problem. It is then clear that any investigation of models of social dynamics involves two levels of difficulty. The first is in the very definition of sensible and realistic microscopic models of interaction betweeb individuals; the second is the usual problem of inferring the macroscopic phenomenology out of the microscopic dynamics of such models.

As we will see with few examples in the next section, most part of the models used in social dynamics are built by defining very simple microscopic laws of interactions between people. This may seem an oversemplification of the real dynamics but in this respect, statistical physics brings an important added value. Indeed, in most situations qualitative (and even some quantitative) properties of large scale phenomena do not depend on the microscopic details of the process. Only higher level features, as symmetries, dimensionality or conservation laws, are relevant for the global behavior. With this concept of universality in mind one can then approach the modelization of social systems, trying to include only the simplest and most important properties of single individuals and looking for qualitative features exhibited by models.

1.2 CONCEPT AND TOOLS OF SOCIAL DYNAMICS: THE ROLE OF THE TOPOLOGY

The fact that people interact with each other spontaneously is probably one of the most self-evident concept in nature, and its importance has been recognized by phylosophyst since Aristotele, who said: "*Man is by nature a social animal; an individual who is unsocial naturally and not accidentally is either beneath our notice or more than human*". Indeed, it seems clear that without interactions, heterogeneity will dominate as each individual in a certain population would follow his personal interests, including opinions, languages and so on so forth. It should be clear, thus, that modeling how the interactions between people happen is the most important aspect when building a social dynamics model. This modeling phase can be conceptually divided into two sub-phases: the definition of the interacting topology and the definition of the interacting laws. The importance of the latter one has already been discussed in the first section of this chapter, hence we will focus our attention on the first one.

It is a common experience that people organize themselves in local communities. This may happen for a number of reasons, for instance two communities might be identified as two groups of people characterized by two different opinions, and in local consenus among the single group (we can think of people who have an Iphone and those who prefer an Android device). However, this is already an high-level effect arising from local interactions, and explaining how these kind of communities emerge is the final result of the model, as we have already explained. Instead, there is another kind of structure that has to be considered as an input to the model: the underlying interacting topology. This is nothing but the definition of relationships among people in a population: who interacts with whom and how frequently. These are the kind of communities that we need to build the model. For instance, one may want to describe the spread of a certain opinion within the population of a certain region; then, the first thing to do would be to define how people are connected each other and we may expect that two close cities share more bonds than cities far from each other. At an higher level of detail, one may be interested in modeling relationships between families, or in general different groups of people that may be linked by common friendships. In this sense, a great example comes from social network structures, like Facebook or Twitter. From a mathematical point of view, the natural way to model relationships of any kind is by means of a *graph*, which is a representation of a network structure made by nodes (people in a population) and links connecting them (relationships between people).

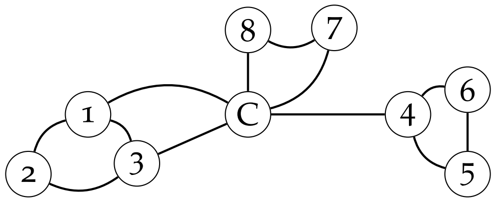


Figure 1: An example of graph representing three small communities indirectly connected each other by the common node C. For instance, this model may represent groups of friends indirectly connected by a "popular" individual, who knows at least a person of each group.

Nowadays, as we have already pointed out, the stunning growth of internet and social networks made the enitre world higly connected by creating huge networks that overcome spatial distances between people. For this reason, the interest is often devoted to the so called *large scale networks*, like the ones we may observe in modern social networks mentioned before.

When observing real networks of this kind, some common features arise:

- The shortest distance between two nodes is usually "small" compared to the network dimension in terms of nodes. This property, known as *small world phenomenom*, was empirically studied by the social psychologist Stanley Milgram, who asked randomly chosen people to try forwarding a letter to a designated "target" person. Roughly, a third of the letters eventually arrived at the target, in a median of six steps (see [4]), which is a surprisingly small number of steps if we think of how many people are present even in a small town.

- Typically there is an over-abundance of hubs - nodes-people in the network with a high number of connections. Notice that this feature is actually linked to the first one as these hubs serve as the common connections mediating the short path lengths between other edges. This property is often analyzed by considering the fraction of nodes in the network that have a particular number of connections going into them (the degree distribution of the network). The formalization of this concept is made by assuming that this degree distribution follows, at leat asimptotically, a power law $p_k \sim k^{-\gamma}$, $2 \leq \gamma \leq 3$ (p_k is the fraction of nodes with degree k).

- These networks are highly clustered. Intuitively, this means that people tend to create tightly knit groups characterised by a relatively high density of ties. In particular, the large number of clusters of order three (triangles) has been often observed. This can be seen as a transitivity property shared by social networks: if a person a knows a person b who in turn knows another person c, then likely there is a tie also between a and c.

Tipically, networks that satisfy these properties are also called *scale-free networks*. There exist several mathematical models to generate graphs with these features, however an extensive discussion of these models is beyond the scope of this dissertation.

1.2.1 *When the graph is complete: the mean field case*

Since our analysis of heterogeneous model will be mainly performed in the mean field assumption, it can be useful to spend few words about the study of social dynamics taking place on a *complete graph*, which is nothing but a graph where every pair of nodes are connected each other. Indeed, in a complete graph all nodes are the same, there are no such things as hubs or communities. Although this may seem an unrealistic framework as we lose the structure characterization of the network, working with a complete graph brings many advantages from a mathematical point of view, which will be discussed in chapter 2. In general, we can say that results obtained under this assumption can provide important indications on how the chosen dynamics work, and say if it is suitable to describe a certain social behaviour. Furthermore, for several models mean field results are coherent with those observed in more general structures like random graphs or they can be used to bound some quantities of interest.

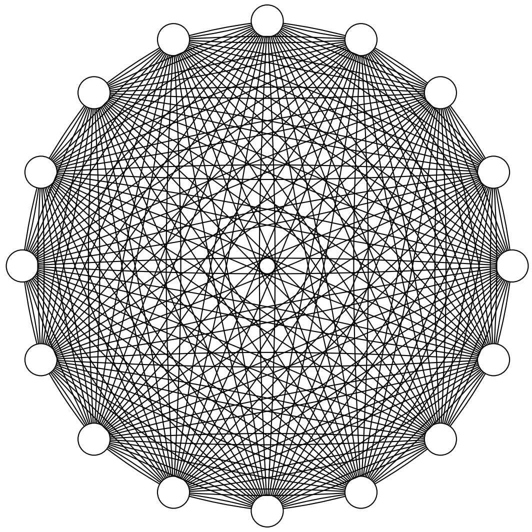


Figure 2: A complete graph with 16 nodes.

1.3 MODELING OPINION DYNAMICS, VOTER AND MAJORITY MODELS

Agreement and disagreement between people are two concepts that we face continously. Everyday life presents many situations in which it is necessary for a group to reach shared decisions. Agreement makes a position stronger, and amplifies its impact on society. An immediate exemplification of this concept can be found in politics, where agreement is crucial in order to make decisions. The main goal of opinion dynamics is to describe how elementary agreement/disagreement rules can influence and change the opinion state of a certain population, leading to the formation of high-level phenomena like the formation of coalitions or consensus.

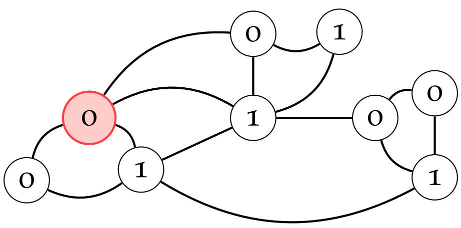
In any mathematical model, opinion has to be a variable, or a set of variables, i.e., a collection of numbers. This may appear too reductive, thinking about the complexity of a person and of each individual position. Everyday life, on the contrary, indicates that people are sometimes confronted with a limited number of positions on a specific issue, which often are as few as two: right/left, Windows/Linux, buying/selling, etc. If opinions can be represented by numbers, the challenge is to find an adequate set of mathematical rules to describe the mechanisms responsible for the evolution and changes of them.

In order to describe how this can be done, we briefly present two models that have received more attention in the literature: the voter and the majority model, that will be used together with their "antisocial" counterpart also in this dissertation.

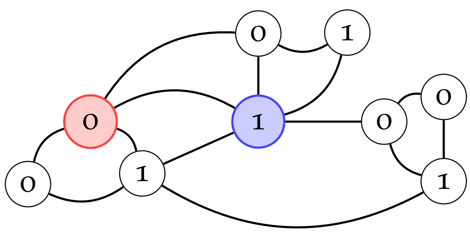
1.3.1 *The voter model*

The voter model brings this name for the very natural interpretation of its rules in terms of opinion dynamics; for its extremely simple definition and intrinsically linear dynamic, the model has been thoroughly investigated also in fields quite far from social dynamics, like probability theory and population genetics. Moreover, many generalizations have been proposed.

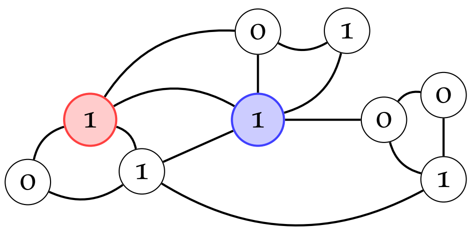
The microscopic dynamic of interaction is extremely simple: each individual in the population is endowed with a binary variable $x \in \{0, 1\}$, which represents its opinion state. At each time step, an individual v is selected along with one of its neighbours w and, with a certain probability q, it copies its opinion: $x_v = x_w$. This update rule implies that people imitate their neighbors. The following pictures show an update step.



(a) The red node is selected for an update.



(b) It looks at a randomly chosen neighbour.



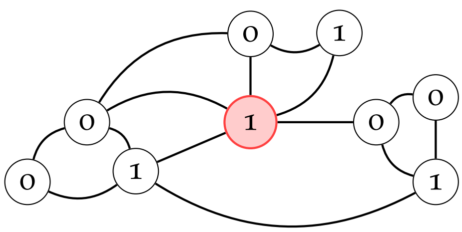
(c) It copies its opinion, in this case 1.

We can already understand that when all sites take the same value, the whole system stops changing forever. Therefore, the voter model has two trivial equilibrium distributions concentrated on the states where all nodes are in state 0 or 1.

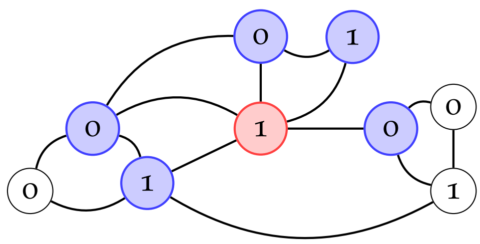
Despite being a rather crude description of any real process, the voter model has soon become popular because it is one of the very few non-equilibrium stochastic processes that can be solved exactly in any dimension.

1.3.2 *The majority model*

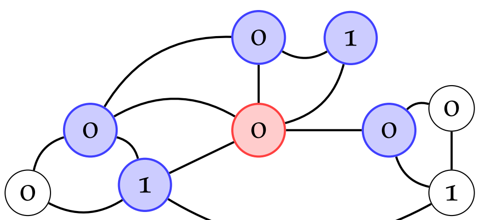
The majority model was proposed to describe public debates and hierarchical voting in a society. Similarly to the voter model, we can assume that each individual is endowed with a binary variable $x \in \{0, 1\}$. The basic principle, as one could guess by the name of the model, is based on a majority rule. In particular, at each instant of time a certain individual is selected and then it looks it copies the local majority of opinion among its neighbours. The majority rule can be stated also imposing that each individual copies the majority opinion of a group of k people randomly chosen among its neighbours; this rule is also known as k-majority. In case of a tie among opinions, one can weather introduce a bias in favor of one of the opinions, say 1, and that opinion prevails in the group, or simply choose one of the two opinions at random.



(a) The red node is selected for an update.



(b) It looks at its neighbours opinions



(c) It selects the local majority, in this case 0.

While the Majority model ignores psycho-sociological aspects of real opinion formation, this simple decision-making process leads to rich collective behavior.

Indeed, this model can be considered a sort of generalization of the plain voter model, and the main difference is that the majority rule considers interactions between multiple individuals, rather than the "1 vs 1" interaction of the voter model (notice that when $k = 1$, the majority model boils down to the voter). This many-body nature of the majority rule makes the model intrinsically non linear and much more complicated to be analized in general networks.

1.4 heterogeneous social dynamics: capturing the diversity

In the first section, we have already pointed out that describing social dynamics by means of sample microscopic laws of interaction can lead to an oversemplification of the model. Indeed, even though interesting high-level patterns arise even from simple models, one should try to include in the model at least the main features that can be observed when analizing social behaviours. One of the most self-evident characteristics of human beings is their enormous behavioural diversity, both within and between populations. People vary in their social, mating and parental behaviour and have diverse and elaborate beliefs, traditions, norms and institutions. These concepts are well explained in [5]. Despite this diversity, most of the social dynamic models in leterature consider all individuals interacting in the same way each other. In other words, once the interaction laws are defined, it is assumed that the whole population follows those laws in the same way. Hence, we can say that all individuals are the same from the point of view of the interaction dynamics, and differences between individuals are only taken into account from a "topological" point of view when defining the network representing a certain population (for instance, the difference between hubs and avarage nodes in a scale-free network). It should appear already clear that working with such *homogeneous* models, like the plain voter and majority model, greatly simplifies the analytical analysis. Neverthless, the assumption of homogeneity seems too restrictive in many cases of interest as it neglects a number of crucial aspects of social behaviours. We always see the effect of this *heterogeneity* among people in our dailylife: people react differently to the same political issue, they can be more or less open-minded towards new opinions, they even may have different behaviours with respect to different kind of people and so on so forth. In particular, an interesting phenomenon that *heterogeneous models* allow to investigate, is the *anti-social behaviour* exhibited by some individuals as they try to alterate the usual spontaneous rute to consensus and bring the system to an highly disordered state, where opinions are fragmentated among the population. Finally, heterogeneous models can be also used to analyze the robustness of the corresponding homogeneous models with respect to perturbations that occur when even a small dynamically heterogeneous group of people arises among the population.

In order to build an heterogeneous model, in principle one should consider different interacting laws for each individual in the population, which of course cannot be done otherwise the model would became uselessy complicated. Moreover, as already mentioned in the introduction paragraph, modeling specific interacting laws for each individual would be an impossible task. The best thing to do in order to overcome these difficulties but without boiling down to an homogeneous model, is to group common diversities by considering *heterogeneous sub-populations* characterized by different interaction dynamics. This heterogeneity can arise in different ways; for instance, individuals of, say, sub-populations 1 and 2 may interact differently from each other with respect to the whole population or rather they might behave in different ways depending on whom they interact with.

In this dissertation we will focus our attention on anti-social behaviours with voter and majority models used as building blocks to construct heterogeneous dynamics. Since the analytical analysis of an heterogeneous model on general networks can easly become unachievable, we will consider mainly dynamics taking place on a complete graph and the model will be studied under mean-field assumptions. However, in the last part of this work we will drop this hypothesis and we will consider an anti-social behaviour model over a star-graph structure.

To being with, in the next chapter we will present some mathematical concepts and tools necessary in order to model and study social dynamics in presence of heterogeneities.

2

MATHEMATICAL CONCEPTS AND TOOLS FOR STOCHASTIC OPINION DYNAMICS

In this chapter we are going to introduce theoretical concepts and tools that will be used throughout this dissertation. The chapter is conceptually divide into two parts: in the first one, we will define all the tools necessary in order to derive and assemble a stochastic opinion dynamics model; in the second part of the chapter, we will present some useful results from dnamic system theory, useful to analyze the models.

More in details, we will introduce a particular class of stochastic process useful to model opinion dynamics on networks, called Markov chains, and their main properties (a more complete analysis of this argument can be found in [7] and [8]). Afterwards, we will present the concept of interacting kernel that will be used to describe how individuals interact each other. The difference between homogenoeus and heterogeneous interacting kernels will be highlighted. Then, we formalize the important and strictly related concepts of mean-field approximation and hydrodynamic-limit, that will allow us to study our models in large networks (mainly in complete graphs). After that, the two important examples of the voter and majority models already presented in the first chapter will be studied by using the introduced tools.

Finally, In the second part we will present some useful results regarding bidimensional differential systems. In particular, we will see theroems about limit sets of planar systems and brief introduction to theory of differential equation with discontinuous right-hand side, which will be useful in the last part of this dissertation.

2.1 DISCRETE TIME MARKOV CHAINS

Markov chains are a special class of stochastic process that, thanks to their "nice" properties, are used as stochastic model of many real-world processes, like opinion and social dynamics.

Definition A *Markov chain* is defined as a set of random variables $\{X_1, X_2, X_3, \dots\}$ defined on a countable set S, called *state space*, such that the following *Markov property* holds:

$$\mathbb{P}(X_{n+1} = x \mid X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \mathbb{P}(X_{n+1} = x \mid X_n = x_n) \quad (1)$$

where the two conditional probabilities are assumed well defined.

We can notice that the Markov property that characterizes Markov chains expresses a sort of memoryless property: the probability distribution of the next state of the process depends only on the current state and not on the sequence of events that preceded it. These kind of processes can thus be used to model systems whose future does not depend on the past but only on the present state of the system itself.

A Markov chain is said to be *time-homogeneous* if $\mathbb{P}(X_{n+1} = x \mid X_n = y) = \mathbb{P}(X_n = x \mid X_{n-1} = y) = P_{yx}$. Intuitively, this means that the transition probability from one state to another does not change in time. In this dissertation we will consider only time-homogeneous Markov chains.

Markov chains are fully described when the transition probabilities are given for any pair of possible states belonging to the state space and together with an inital discrete probability distribution defined on it, which is the initial condition of the chain. Tipically, the transition probabilities are collected in a stochastic matrix $P \in \mathbb{R}^{S \times S}$, called *transition matrix*, while the initial distribution is represented as a probability vector usually denoted by $\pi(0) \in \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the set of all vectors in $[0,1]^S$ whose entries add up to 1. Notice that $\pi(0)$ represents the probability distribution of the variable X_0 and if the chain starts deterministically from a certain state x, then $\pi(0) = \delta(x)$. The probability distribution $\pi(1)$ of the state after one time step can be found by means of the transition matrix: $\pi(1) = P \cdot \pi(0)$. If we iterate this relation, we easly see that the probability distribution of the state of the system after n steps, which is the random variable X_n, is given by:

$$\pi(n) = P^n \cdot \pi(0)$$

Let us see some other properties that a Markov chain might feature.

Let i, j be two states belonging to S, then i is said to be reachable from j when a system started in state j has a non-zero probability of transitioning into state i at some point, that is, there exists $k \in \mathbb{N}$ such that $\mathbb{P}(X_{n+k} = i \mid X_n) > 0$. A Markov chain is called *irreducible* if each state is reachable from every other state. This property is particularly nice as it is possible to prove that an finite irreducible Markov chain admits a unique probability distribution $\pi \in \mathcal{P}(S)$ such that if a system starts with initial condition π-distributed, then also the distributions of the next states will have the same distribution π. More formally, π is such that if X_n is π-distributed, then X_{n+1} is π-distributed as well. This ditribution is called the *invariant probability* or invariant distribution of the chain. Recalling that we can always write $\pi(n+1) = P \cdot \pi(n)$, it immediately follows that the invariant probability must satisfy the equation

$$\pi = P \cdot \pi \tag{2}$$

which shows that π is nothing but a left eigenvector associated to the eigenvalue 1 of the transition matrix P. In particular, when the chain is irreducible, the Perron-Frobenius theorem guarantees the existence, the uniqueness and the non-negativity of such eigenvector.

A Markov chain is said to be *aperiodic* if there exists a $\overline{n} \in \mathbb{N}$ such that $\forall\, \overline{n} > n$ we have:

$$\mathbb{P}(X_{\overline{n}} = i \,|\, X_0 = i) > 0 \qquad \forall\, i \in S \tag{3}$$

The concept of aperiodicity means that returns to state i can occur at irregular times.

A Markov chain over a finite state space S is said to be *ergodic* if it is both irreducible and aperiodic.

The following important result holds for ergodic Markov chains:

Proposition 2.1.1 *Let X_n be an ergodic Markov chain over a finite state space S and π its unique invariant probability. Then for all initial distributions $\pi(0)$ the following holds:*

$$\lim_{n\to\infty} \pi(n) = \pi \tag{4}$$

Another important concept regarding Markov chains is the *reversibility*. A Markov chain is said to be reversible if there is a probability distribution over states $\tilde{\pi}$, such that the following holds:

$$\tilde{\pi}_i \mathbb{P}(X_{n+1} = j \,|\, X_n = i) = \tilde{\pi}_j \mathbb{P}(X_{n+1} = i \,|\, X_n = j) \qquad \forall\, i, j \in S \tag{5}$$

This condition is usually know as *detailed balance*. A result proven in [bibl] shows that $\tilde{\pi}$ is an invariant probability for the chain.

Another feature of Markov chains is based on the so called *absorbing states*. A state i is said to be absorbing if it is impossible to leave such a state, that is, $P_{ii} = 1$. A Markov chain is called *absorbing Markov chain* if there for any state j there is at least a reachable absorbing state i. If we denote with $\mathcal{A}$ the set of the absorbing states of an absorbing Markov chain, then the following holds:

$$\lim_{n\to\infty} \mathbb{P}(X_n \in \mathcal{A} \,|\, X_0 = i) = 1 \qquad \forall\, i \in S \tag{6}$$

This means that, in an aborbing Markov chain, soon or later one will jump into one of the absorbing states and it will remain stuck there forever, regardless of the initial condition. Notice, however, that such chains are not ergodic and the above statement does not specify which one of the absorbing states the system will converge to as this depends in general on the initial condition.

Because of their definition in terms of transition probabilities from a state to another, Markov chains can be visually represented by a directed *graph*. We have already seen what is a graph in the first chapter, ate least intuitively. In more formal words, a graph $\mathcal{G}$ is a mathematical object defined by a set of nodes (or vertices) $\mathcal{V}$ and a set of links (or edges) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ connecting them, and it is usually denoted with $\mathcal{G}(\mathcal{V}, \mathcal{E})$. A graph may be directed or undirected. We say that a graph is directed if $\forall\, (i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E}$.

(a) Directed graph. (b) Undirected graph.

Given a time-homogeneous Markov chain, it is possible to define its underlying graph by means of its transition matrix. In particular, one can define a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ such that $\mathcal{V} = S$ and $(i, j) \in \mathcal{E} \iff P_{ij} > 0$. The interpretation of this construction is quite obvious: the outgoing arrows at a certain state (a node of the graph) are nothing but the possible directions for a jump at the next time step. Notice that if the Markov chain is not time-homogeneous, one should define a sequence of graphs, one for each time step as the transition probabilities may vary in time. More in general, it is possible to construct an underlying graph even for non markovian processes by using the same logic.

Usually, the transition probability values are reported near the edges. Let us see a simple example of a simple Markov chain.

Example: weather evolution Here we will consider a Markov chain describing a very simple model of weather evolution. Let us assume that a study of the weather in the city of Ottawa in early spring yields the following observations:

- It is almost impossible to have two nice days in a row
- If we have a nice day, we just as likely to have snow or rain the next day
- If we have snow or rain, then we have an even chance to have the same the next day
- If there is a change from snow or rain, only half of the time is this a change to a nice day.

Notice that this toy model assumes that the probability of having a certain weather condition tomorrow depends only on the weather of today.

We can thus model the weather dynamics by means of a Markov chain with state space $S = \{\text{nice day, rain, snow}\} = \{\text{n, r, s}\}$. Indeed, by using the above informations, we can easly write the transition matrix of the chain

$$P = \begin{bmatrix} 0 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.25 \\ 0.5 & 0.25 & 0.5 \end{bmatrix} \tag{7}$$

and the corresponding underlying graph representing the chain is the following:

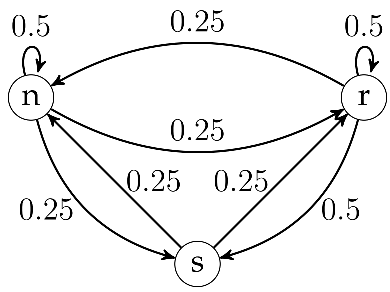


Figure 3: The underlying graph representing the weather evolution model.

Notice that the Markov chain considered in this example is ergodic, and thus it admits a unique invariant probability that coincides with the asymptotic probability to find a certain weather condition in Ottawa. Applying equation (2) one may find that $\pi = [0.2, 0.4, 0.4]^T$, which means that, in the long term, there is 20% chance of getting a nice day, 40% chance of having a rainy day and 40% chance of having a snowy day.

2.2 INTERACTION KERNELS

In this section we will introduce some mathematical objects that allow us to model interactions between individuals in opinion dynamics models, called *interaction kernels*.

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a strongly connected undirected graph with finite sets of nodes $\mathcal{V}$ and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Let $|\mathcal{V}| = N$ be the population. In the following nodes will be often called individuals, whenever more appropriate for the applicative scenario we have in mind; edges represent communication capabilities and relationships between them.

Given a certain node v in the graph, its *neighbourhood* $\mathcal{N}_v$ is defined as the set of nodes the node itself is connected to:

$$\mathcal{N}_v := \{w \in \mathcal{V} \mid (v, w) \in \mathcal{E}\} \tag{8}$$

Notice that the degree of v is: $d_v = |\mathcal{N}_v|$. Let $\mathcal{X}$ be a finite set representing all the possible states, in this case opinions, that a certain individual can have. In particular, we shall denote with $x_v(t) \in \mathcal{X}$ the opinion (or state) assumed by individual v at time t. All these states can be collected in a large vector $\mathbf{x}(t) \in \mathcal{X}^{\mathcal{V}}$, which is called *detailed configuration* of the system at time t.

A generic stochastic dynamics over this network is defined as follows: at each time instant t, nodes, independently from each other, activate and their state possibly changes, in a probabilistic way, depending on the states of their neighbours- In order to model how this actually happens, we introduce a family of functions Θ_v called interaction kernels:

$$\Theta_v \,:\, \mathcal{X} \times \mathcal{X} \times \mathcal{X}^{\mathcal{N}_v} \to [0,1] \tag{9}$$

where $\Theta_v(x_v, x'_v \,|\, \mathbf{x})$ is the probability that node v changes its state from x_v to x'_v conditioned to the fact that the detailed configuration of its neighbours is x. Notice that obviousy the following holds:

$$\sum_{x' \in \mathcal{X}} \Theta_v(x_v, x'_v \,|\, \mathbf{x}) = 1 \qquad \forall\, v \in \mathcal{V},\, \forall\, \mathbf{x} \in \mathcal{X}^{\mathcal{N}_v}, \quad x \in \mathcal{X} \tag{10}$$

Interaction kernels induce naturally a Markov chain over the space of configuration $\mathcal{X}^{\mathcal{V}}$. Indeed, since at each instant of time only one node can activate and thus change opinion, transitions can only be possible between detailed configurations that differ by not more than one element (the updated node at that instant of time). Hence, given two detailed configurations x and x′, the transition probabilities that characterize the Markov chain can be written as follows:

$$P_{\mathbf{x},\mathbf{x}'} = \begin{cases} 0 & \text{if} \quad H_d(\mathbf{x},\mathbf{x}') > 1 \\ \rho_v \Theta_v(x_v, x'_v \,|\, \mathbf{x}) & \text{if} \quad H_d(\mathbf{x},\mathbf{x}') = 1 \\ 1 - \sum_{\mathbf{x} \neq \mathbf{x}'} P_{\mathbf{x},\mathbf{x}'} & \text{otherwise} \end{cases} \tag{11}$$

Here H_d denotes the Hamming distance[1] and ρ_v is the probability that node v is activated at time t.

A special kind of interaction kernels are the so called *gossip interaction kernel*. A kernel is said to be of gossip type if there exists a stochastic matrix $L \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ and a family of functions (called gossip interacting functions)

$$\theta_{vw} \,:\, \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0,1] \tag{12}$$

1 The Hamming distance between two vectors is defined as the distance induced by the Hamming norm, which is the number of non-zero elements in a vector. Hence, the Hamming distance between two vectors can be seen as the number of elements in which the two vectors differ.

such that $\sum_{x' \in \mathcal{X}} \theta_{vw}(x, x' \mid y) = 1$ for every $x, y \in \mathcal{X}$, and

$$\Theta_v(x_v, x'_v \mid \mathbf{x}) = \sum_{w \in \mathcal{N}_v} L_{vw} \theta_{vw}(x_v, x'_v \mid x_w) \tag{13}$$

In words, in the dynamics defined through a gossip interaction kernels, after the selection of the node v according to the probability vector ρ, a second node w is selected according to the probabilities given by the stochastic matrix L. Then, the probability of an opinion shift of the node v from x_v to x'_v, conditioned on the state of the second node x_w, is given by $\theta_{vw}(x, x' \mid x_w)$. In the case when the functions $\theta_{vw}(x, x' \mid x_w)$ do not explicitly depend on the present opinion of v, the interaction kernels are said to be a *forgetful gossip*.

Furthermore, if the following conditions hold:

- $\rho = \frac{1}{|\mathcal{V}|}$, which means that nodes are selected uniformy at random.
- L defines a simple random walk on $\mathcal{G}$, which means that the neighbours w are also selected at random.
- θ_{vw} does not explicitely depend on v and w.

then we talk about *homogeneous* gossip interactive dynamics and the subscript in θ_{vw} can be dropped. Notice that in this case, the elements of the transition matrix P for which $H_d(\mathbf{x}, \mathbf{x}') = 1$ are given by:

$$P_{\mathbf{x},\mathbf{x}'} = \sum_{w \in \mathcal{N}_v} \frac{1}{|\mathcal{V}|} \frac{1}{d_v} \theta(x_v, x'_v \mid x_w) \tag{14}$$

In principle, one may define different interacting kernels depending not only on the activated node v, but also on different groups of nodes involved in the interaction, as an individual may behave differently with different people. In order to address for these more general situations, one may consider the global population of nodes $\mathcal{V}$ splitted into a finite number of subpopulations with different interactive attitudes and thus different dynamic behaviors.

More in details, we partition the population as follows:

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \qquad \mathcal{V}_i \cap \mathcal{V}_j = \emptyset \quad \forall\, i \neq j \tag{15}$$

We have that $|\mathcal{V}_i| = N_i$ is the population size of the i-th subpopulation (obviously $N = N_1 + N_2 + \cdots + N_s$). Then, fixed a state space $\mathcal{X}$, we define the set of *heterogeneous* interaction kernels as the family of functions

$$\Theta_v \,:\, \mathcal{X} \times \mathcal{X} \times \prod_{j=1}^{s} \mathcal{X}^{\mathcal{V}_j} \to [0, 1] \tag{16}$$

Thus, the probability that a node v changes its opinion from x_v to x'_v conditioned to the detailed configurations of *each* subpopulation is given by:

$$\mathbb{P}(x_v \to x'_v \,|\, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = \Theta_v(x_v, x'_v \,|\, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) \tag{17}$$

Notice the explitic dependence on the different kind of groups present in the population.

2.3 MEAN FIELD APPROXIMATION

The markov chain induced by the interaction kernels is defined over the generally huge space of detailed configurations $\mathcal{X}^{\mathcal{V}}$, which grows exponentially in the number of nodes. Moreover, the transition probabilities of the chain depend on the graph tolpology as the interactive kernels structure is coupled with it. To analyze a model in this general framework can be exceedingly difficult. For this reasons, it would be nice to work with a sort of equivalent Markov chain over a smaller state space. This is possible by means of the so called *mean field approximation*. This approach usually assumes that the underlying network is the complete graph.

From the mathematical point of view, the key point is that the mean field assumption allows to take a Eulerian point of view and instead of keeping track of the state of each single agent we can rather consider the fraction of agents sharing a certain state. This remarkable semplification has two main effect: it makes possible to reduce the dimension of the state space drastically and allows us to consider a kind of *hydrodynamic limit* when the population is large, that is, $N \to \infty$.

Let us fix some notations. We recall that $\mathcal{P}(\mathcal{X})$ is the set of probability vector over $\mathcal{X}$, that is

$$\mathcal{P}(\mathcal{X}) = \left\{ p \in \mathbb{R}^{\mathcal{X}} \,|\, p_x \geq 0, \sum_x p_x = 1 \right\} \tag{18}$$

Furthermore, we define

$$\mathcal{P}_N(\mathcal{X}) = \{ p \in \mathcal{P}(\mathcal{X}) \,|\, Np \in \mathbb{N}^{\mathcal{X}} \} \tag{19}$$

Given a detailed configuration of the system $\mathbf{x} \in \mathcal{X}^{\mathcal{V}}$, it is possible to define its *type* as the probability vector $p^{\mathbf{x}} \in \mathcal{P}(\mathcal{X})$, whose components are defined by:

$$p_i^{\mathbf{x}} = \frac{\#\{v \in \mathcal{V} \,|\, x_v = i\}}{|\mathcal{V}|} \qquad \forall\, i \in \mathcal{X} \tag{20}$$

We see that the generic component $p^{\mathbf{X}}$ is the fraction between the number of individuals with opinion equal to i in the configuration $\mathbf{x}$ and the total population size. Notice that $p_i^{\mathbf{x}} \in \mathcal{P}_N(\mathcal{X})$ and the space $\mathcal{P}_N(\mathcal{X})$ is tipically called the *space of types*.

In the following, we will introduce the mean field approach firstly for homogeneous interaction kernels; afterwards, we will extend the same concept to heterogeneous kernels of the form (16).

To begin with, we fix an interaction kernel $\Theta : \mathcal{X} \times \mathcal{X} \times \mathcal{X}^{\mathcal{V}} \to [0,1]$, which is the same for every node. Notice that we are assuming that $\mathcal{N}_v = N$, which is true for complete graphs. Also, this means that each individual is a neighbour of itself: this does not entail any loss of generality and it has the advantage that the kernel has the same domain for all individuals.

Definition The interaction kernel Θ is said to be an *anonymous interaction kernel* if $\Theta(x, x' \,|\, \mathbf{x})$ is invariant by all possible permutations of the vector $\mathbf{x}$.

Equivalently, we can say that Θ is anonymous if there exists a function $\Pi : \mathcal{X} \times \mathcal{X} \times \mathcal{P}_N(\mathcal{X}) \to [0,1]$ such that the following holds:

$$\Theta(x, x' \,|\, \mathbf{x}) = \Pi(x, x' \,|\, p^{\mathbf{x}}) \qquad \forall\, \mathbf{x} \in \mathcal{X}^{\mathcal{V}}, \quad x, x' \in \mathcal{X} \tag{21}$$

In words, the above definition states that an anonymous interaction kernel is well defined when the type of the system is known. In particular, the detailed configuration does not matter and it is sufficient to know how many different opinions (or states) are present among the population; how these opinions are located in the network topology does not matter. Notice that is always true when we work with complete graphs.

As we have already anticipated, working in complete graphs and thus with anonymous kernels, greatly simplifies the analysis of the model. Indeed, if $X(t) = \mathbf{x}(t)$ is the Markov process governing the evolution of the detailed configuration by means of the interaction kernel Θ, one can use the equality (21) to define the corresponding process of types

$$R(t) := p^{X(t)} \tag{22}$$

Notice that the so defined $R(t)$ is the projection of the process $X(t)$ onto the space of types $\mathcal{P}_N(\mathcal{X})$. A remarkable fact is that $R(t)$ is also a jump Markov process on $\mathcal{P}_N(\mathcal{X})$ whose transition probabilities can be described as follows. When a jump occurs in the original Markov process $X(t)$, it means that an individual has changed its opinion from some $x \in \mathcal{X}$ to some $x' \in \mathcal{X}$. This corresponds to a change in exactly two components of the type process $R(t)$ jumping from some p to $p' = p + \frac{1}{N}(\delta^{x'} - \delta^{x})$ [2]. Infact, when $x \to x'$, we have that the fraction of opinion x decreases by $\frac{1}{N}$ while the fraction of x' increases by the same quantity.

2 δ^x is the the vector made by all zeros and a 1 in position x, namely: $\delta_i^x = 0\,\forall i \neq x$ and $\delta_i^x = 1$ for $i = x$.

The transition probabilities associated to the process $R(t)$ will be deonted with $Q_{p,p'}$ and it clearly holds that:

$$Q_{p,p'} = \begin{cases} p_x \Pi(x, x' \mid p) & \text{if} \quad H_d(p, p') = 2 \\ 1 - \sum_{p \neq p'} p_x \Pi(x, x' \mid p) & \text{if} \quad H_d(p, p') = 0 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

It is important to highlight the fact that, while we can always consider the process $R(t)$ even when the underlying graph is not complete, such a process will not be Markovian in general. How opinions are distributed in the population is not a sufficient statistics to determine the evolution of the system as an agent will be influenced only by a subset of agent where, in general, opinions distribution will be different from the global one.

That said, we can observe that the Markov process $X(t)$ is defined over the space of detailed configurations, which has a cardinality that grows exponentially with the number of nodes $|\mathcal{V}| = N$; conversely, the new process $R(t)$ is defined over the space of types whose cardinality grows only polynomially [3] with N. This yields to a great semplification in the model analysis.

The definition of anonymous kernel can be extended to heterogeneous kernels of the form (16), as well as the concept of projected process onto the space of types.

Definition Assume that the set of nodes $\mathcal{V}$ is decomposed into subpopulations according to (15). Then the heterogeneous interaction kernel Θ_v is said to be a *semi-anonymous interaction kernel* if there exist functions

$$\Pi_k \, : \, \mathcal{X} \times \mathcal{X} \times \prod_{j=1}^{s} \mathcal{P}_{N_j}(\mathcal{X}) \to [0, 1] \tag{24}$$

for $k = 1, \ldots, s$, such that, if $v \in \mathcal{V}_k$, it holds:

$$\Theta_v(x, x' \mid \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s) = \Pi_k(x, x' \mid p^{\mathbf{x}_1}, p^{\mathbf{x}_2}, \ldots, p^{\mathbf{x}_s}) \qquad \forall \, \mathbf{x} \in \mathcal{X}^{\mathcal{V}}, \; x, x' \in \mathcal{X} \tag{25}$$

Like we did before, let $X(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_s(t))$ be the Markov process governing the evolution in time of the detailed configurations by means of the interaction kernels Θ_v. Then, one can define the corresponding process of types $R(t) = (R_1(t), R_2(t), \ldots, R_s(t))$ with generic element $R_k(t) = p^{X_k(t)}$, that is the type process relative to the individuals belonging to subpopulation $\mathcal{V}_k$.

3 In particular, one may prove that: $|\mathcal{P}_N(\mathcal{X})| \leq (N - |\mathcal{X}| - 1)^{(|\mathcal{X}|-1)}$

We have that $R(t)$ is still a jump Markov process on the space $\prod_{j=1}^{s} \mathcal{P}_{N_j}(\mathcal{X})$ characterized by transition probabilities that can be defined as follows. Let us fix the notation $R(t) = \mathbf{p} = (p^1, p^2, \dots, p^s)$. When a jump occurs in the original Markov process $X(t)$, it means that an individual of a certain subpopulation $v \in \mathcal{V}_k$ has changed its opinion from some $x \in \mathcal{X}$ to some $x' \in \mathcal{X}$. This corresponds to a change in exactly two components of the type process $R_k(t) = p^k$. Thus, the types process $R(t)$ jumps from some $\mathbf{p}$ to $\mathbf{p}' = \mathbf{p} + \frac{1}{N_k}(\delta^{k,x'} - \delta^{k,x})$ [4]

The associated transition probabilities are thus given by:

$$Q_{\mathbf{p},\mathbf{p}'} = \frac{N_k}{N} p_x^k \Pi_k(x, x' \mid \mathbf{p}) \tag{26}$$

In words, the probability of jumping from $\mathbf{p}$ to $\mathbf{p}'$ as effect of an opinion shift of a node $v \in \mathcal{V}_k$ from x to x', is found by multipling the probability of selecting a node belonging to $v \in \mathcal{V}_k$ by the probability of picking up a node with opinion x by the probability of this transition given by the interaction kernel.

2.4 THE HYDRODYNAMIC LIMIT

The mean field approximation allows us to study the model for large networks, that is, $N \to \infty$. An important result, known as *Kurts's theorem*, will be provided and will show that, in this regime, the model can be descripted by means of deterministic differential equations. We will start discussing results for homogeneous kernels; afterwards, the theory will be extended in order to take into account heterogeneous kernels.

2.4.1 *Hydrodynamic limit for homogeneous models*

Since all the transition probabilities depend on the number of individuals N, we will emphasize this dependence by adding a superscript (N) when necessary. In particular, the interaction kernels and the type process will be denoted with $\Pi(x, x' \mid p)^{(N)}$ and $R^{(N)}(t)$ respectively.

4 $\delta^{k,x} \in \prod_{j=1}^{s} \mathcal{P}_{N_j}(\mathcal{X})$ takes value 1 in position x of the k-th subvector.

We start noticing the following

$$\begin{aligned}\mathbb{E}[R^{(N)}(t+1) \mid R^{(N)}(t)] &= R^{(N)}(t) + \frac{1}{N}\sum_{x'\in\mathcal{X}} \mathbb{P}(R^{(N)}(t+1) = p + \frac{1}{N}(\delta^{x'} - \delta^{x})|p)\delta^{x'} \\ &- \frac{1}{N}\sum_{x'\in\mathcal{X}} \mathbb{P}(R^{(N)}(t+1) = p + \frac{1}{N}(\delta^{x} - \delta^{x'})|p)\delta^{x} \\ &= R^{(N)}(t) + \frac{1}{N}\sum_{x,x'\in\mathcal{X}} Q^{(N)}_{p(t),p'}(\delta^{x'} - \delta^{x})\end{aligned} \tag{27}$$

Now we define the so called *drift operator* $\mathcal{F} : \mathcal{P}^{(N)}_{N}(\mathcal{X}) \to \mathbb{R}^{\mathcal{X}}$

$$\mathcal{F}^{(N)}(R^{(N)}(t)) := \sum_{x,x'\in\mathcal{X}} Q^{(N)}_{p(t),p'}(\delta^{x'} - \delta^{x}) \tag{28}$$

Thus, equation (27) can be re-written as in terms of $\mathcal{F}^{(N)}$:

$$\mathbb{E}[R^{(N)}(t+1) \mid R^{(N)}(t)] = R^{(N)}(t) + \frac{1}{N}\mathcal{F}^{(N)}(R^{(N)}(t)) \tag{29}$$

but this implies that

$$R^{(N)}(t+1) = R^{(N)}(t) + \frac{1}{N}\mathcal{F}^{(N)}(R^{(N)}(t)) + \frac{1}{N}w(t) \tag{30}$$

where $w(t)$ is a random process taking value on $\mathbb{R}^{\mathcal{X}}$ and such that its expectation conditioned on $R^{(N)}(t)$ is the 0-vector, that is, $\mathbb{E}[w(t) \mid R^{(N)}(t)] = 0$.

Now, let us change the time scale by considering $t = \lfloor \tau N \rfloor$ and we consider the processes $\tilde{R}^{(N)}(\tau) = R^{(N)}(\lfloor \tau N \rfloor) = R^{(N)}(t)$ and $\tilde{w}(\tau) = w(\lfloor \tau N \rfloor) = w(t)$. Intuitively, this rescaling should make sense as for each discrete time step the type process makes movements of magnitudo $\frac{1}{N}$. Thus, in order to appreciate the dynamics we need a time scaling with N at least.

Therefore, equation (30) becomes

$$\tilde{R}^{(N)}(\tau + \frac{1}{N}) = \tilde{R}^{(N)}(\tau) + \frac{1}{N}\mathcal{F}^{(N)}(\tilde{R}^{(N)}(\tau)) + \frac{1}{N}\tilde{w}(\tau) \tag{31}$$

which is equivalent to

$$\frac{\tilde{R}^{(N)}(\tau + \frac{1}{N}) - \tilde{R}^{(N)}(\tau)}{\frac{1}{N}} = \frac{1}{N}\mathcal{F}^{(N)}(\tilde{R}^{(N)}(\tau)) + \frac{1}{N}\tilde{w}(\tau) \tag{32}$$

If now we let $N \to \infty$, we may observe that the left-hand side of equation (32) converges to the derivative of $\tilde{R}^{(N)}(\tau)$ and the equation seems to become a differential equation. This intuition can indeed be made precise adding some extra but mild conditions. It is the content of the following important result, whose proof can be found in [6].

Theorem 2.4.1 (Kurtz's theorem (homogeneous models)) *Suppose that:*

- *For* $N \to \infty$ *we have* $\Pi^{(N)}(x, x', p) \to \Pi(x, x', p)$ *uniformly in* p*, namely:*

$$\lim_{N \to \infty} \sup_{p \in \mathcal{P}_N(\mathcal{X})} \left\| \Pi^{(N)}(x, x', p) - \Pi(x, x', p) \right\| = 0 \tag{33}$$

- *For* $N \to \infty$ *we have that* $R^{(N)}(0)$ *converges to a probability vector* $\eta_0 \in \mathcal{P}(\mathcal{X})$*.*

- *For* $N \to \infty$ *we have that the drift operator* $\mathcal{F}^{(N)}$ *converges uniformly to an operator* $\mathcal{F} : \mathcal{P}(\mathcal{X}) \to \mathbb{R}^{\mathcal{X}}$ *which is Lipschitz continuous.*

  Namely, $\forall\, \varepsilon > 0,\ \exists\, N_0 \in \mathbb{N}$ *such that:*

$$\left\| \mathcal{F}^{(N)}(p) - \mathcal{F}(p) \right\| < \varepsilon \qquad \forall\, p \in \mathcal{P}_N(\mathcal{X}),\ \forall\, N \geq N_0 \tag{34}$$

  where $\mathcal{F}$ *is Lipschitz continuous.*

Let $\eta(t)$ *be the unique solution of the following Cauchy problem:*

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = \mathcal{F}(\eta) \\ \eta(0) = \eta_0 \end{cases} \tag{35}$$

Then, for any fixed $T > 0$*, there exists a constant* $C_T > 0$ *such that, for every* $\epsilon > 0$*, we have:*

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |R^{(N)}(t) - \eta(t)| > \epsilon \right) \leq 2|\mathcal{X}|^2 \exp(-C_T N \epsilon^2) \tag{36}$$

In particular, almost surely

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T} |R^{(N)}(t) - \eta(t)| = 0 \tag{37}$$

This fundamental theorem states that, when N is sufficiently large, the dynamics of the model can be described by means of the deterministic system of differential equations (35). Notice that since $\mathbb{1} * \mathcal{F}(\eta) = 0$ [5], we have that $\mathbb{1} * \eta(t)$ is invariant on the trajectories of the ODE (35). Considering that the initial condition is a probability vector, we have that $\eta(t) \in \mathcal{P}(\mathcal{X})$ for all t. In particular, this implies that we can reduce the dimension of the ODE to $|\mathcal{X} - 1|$.

An important special case is when we are dealing with a binary state space $\mathcal{X} = \{0, 1\}$. In this case, equation (35) boils down to a scalar ODE. Indeed, we just

5 $\mathbb{1}$ is the vector of all ones.

need the fraction of 1's in the population as variable to keep track of the process. We call this fraction $z^N(t)$. Notice that the underlying markov chain becomes a birth and death process with transition probabilities $q^{+N}(z)$ and $q^{-N}(z)$, that are respectively the probabilities to increase or reduce the fraction of ones by $\frac{1}{N}$ when this fraction is $z^N(t)$. Assuming that when $N \to \infty$ they uniformly converge to two Lipschitz-continuous functions $q^+(z)$ and $q^-(z)$, Kurtz's theorem applies and the corresponding 1-dimensional ODE is given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = q^+(z) - q^-(z) \tag{38}$$

Notice that equation (38) is a sort of probability balance equation, whose right-hand side is the difference between the mean fraction of individuals changing their opinion from 0 to 1 and the mean of those making the change in the opposite direction.

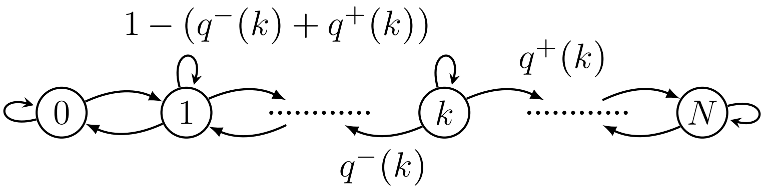


Figure 4: A birth and death chain with state space $S = \{0, 1, \ldots, N\}$

2.4.2 *Hydrodynamic limit for heterogeneous models*

Hydrodynamic limit can be studied also in presence of subpopulations following different dynamics. Considering the partition (15), the hydrodynamic limit takes the form $N = N_1 + N_2 + \cdots + N_s \to \infty$, under the assumption that:

$$\frac{N_k}{N} \to \rho_k \in [0,1] \qquad \text{for } k = 1, \ldots, s \tag{39}$$

One can then apply the same argument we have followed from equation (27). In particular, we can define the *drift operator*: $\mathcal{F}^{(N_1,\ldots,N_s)} : \prod_{j=1}^{s} \mathcal{P}_{N_j}(\mathcal{X}) \to \mathbb{R}^{(\mathcal{X})^s}$ as:

$$\mathcal{F}^{(N_1,\ldots,N_s)}(R^{(N_1,\ldots,N_s)}(t)) := \sum_{k=1}^{s} \sum_{x,x' \in \mathcal{X}} Q^{(N_1,\ldots,N_s)}_{k,p(t),p'}(\delta^{k,x'} - \delta^{k,x}) \tag{40}$$

Hence, we can present the following extended version of the kurt's theorem.

Theorem 2.4.2 (Kurtz's theorem (heterogeneous models)) *Suppose that:*

- *For* $N \to \infty$ *we have* $\Pi_k^{(N_1,\dots,N_s)}(x, x', \mathbf{p}) \to \Pi_k(x, x', \mathbf{p})$ *uniformly in* $\mathbf{p}$
- *For* $N \to \infty$ *we have that* $R^{(N_1,\dots,N_s)}(0))$ *converges to a probability vector* $\eta_0 \in \mathcal{P}(\mathcal{X})^s$.
- *For* $N \to \infty$ *we have that the drift operator* $\mathcal{F}^{(N_1,\dots,N_s)}$ *converges uniformly to an operator* $\mathcal{F} : \mathcal{P}(\mathcal{X})^s \to (\mathbb{R}^{\mathcal{X}})^s$ *which is Lipschitz continuous.*

Let $\eta(t)$ *be the unique solution of the following Cauchy problem:*

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = \mathcal{F}(\eta) \\ \eta(0) = \eta_0 \end{cases} \tag{41}$$

Then, for any fixed $T > 0$*, there exists a constant* $C_T > 0$ *such that, for every* $\epsilon > 0$*, we have:*

$$\mathbb{P}\left(\sup_{0 \le t \le T} |R^{(N_1,\dots,N_s)}(t) - \eta(t)| > \epsilon \right) \le 2|\mathcal{X}|^2 \exp(-C_T N \epsilon^2) \tag{42}$$

In particular, almost surely

$$\lim_{N \to \infty} \sup_{0 \le t \le T} |R^{(N_1,\dots,N_s)}(t) - \eta(t)| = 0 \tag{43}$$

Notice that since $\mathbb{1}^k * \mathcal{F}(\eta) = 0$ [6], we have that $\mathbb{1}^k * \eta(t)$ is invariant on the trajectories of the ODE (41). Considering that the initial condition is in $\mathcal{P}(\mathcal{X})^s$, we have that $\eta(t) \in \mathcal{P}(\mathcal{X})^s$ for all t. In particular, this implies that we can reduce the dimension of the ODE to $s|\mathcal{X} - 1|$.

2.4.3 *An important case: binary state space*

In the special case when $\mathcal{X} = \{0, 1\}$ is binary, we obtain a s-dimensional system of ODE. Indeed, in this case, considering the fraction z_k of 1's in the subpopulation $\mathcal{V}_k$, we obtain a s-dimensional birth and death process where the admissible transitions are in the increase or decrease of one of the z_k of the quantity $\frac{1}{N_k}$ with corresponding probabilities denoted by $q_k^{+(N_1,\dots,N_s)}(\mathbf{z})$ and $q_k^{-(N_1,\dots,N_s)}(\mathbf{z})$ (where $\mathbf{z} = (z_1, \dots, z_s)$). Assuming that when $N \to \infty$ they uniformly converge to two Lipschitz-continuous functions $q_k^+(\mathbf{z})$ and $q_k^-(\mathbf{z})$, Kurtz's theorem applies and the

6 $\mathbb{1}^k$ is the vector consisting of all 0 subvectors, but the k-th subvector which consists of all 1's.

generic k-th equation of the corresponding s-dimensional system of ODE is given by

$$\frac{\mathrm{d}z_k}{\mathrm{d}t} = q_k^+(\mathbf{z}) - q_k^-(\mathbf{z}) \tag{44}$$

In this dissertation we will consider heterogeneous opinion models considering two subpopulations, that is, $s = 2$ and $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$. This further semplification reduces the Markov process to a bidimensional birth and death process and the system of ODE to a bidimensional differential system. In the following and in the rest of this work we will call x the fraction of ones in the first subpopulation $\mathcal{V}_1$ and y the fraction of ones in $\mathcal{V}_2$. Namely:

$$x = \frac{n_1}{N_1} \qquad y = \frac{n_2}{N_2} \tag{45}$$

where n_1 and n_2 are the number of ones of population 1 and 2 respectively. Notice that we have $x, y \in [0,1]^2$. Furthermore, if we define $\rho_1 = N_1/N$ and $\rho_2 = N_2/N$ as the fractions of the two populations, then the total fraction of ones is obviously given by $\frac{n}{N} = \rho_1 x + \rho_2 y$.

Hence, the planar differential system describing the hydrodynamic limit dynamics takes the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = q_1^+(x,y) - q_1^-(x,y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = q_2^+(x,y) - q_2^-(x,y) \end{cases} \tag{46}$$

Working with bidimensional systems like (45) allows us to use the powerful tools from the theory of planar autonomous dynamical systems. In the next section of the chapter we are going to see some of those.

2.5 bidimensional differential systems: some useful results

In this section we are going to show some results from dynamical system theory that will be used to study our models. A systematic analysis of this topic can be found in [15].

The first result is the fundamental *Poincare-Bendixon theorem*, which completely characterizes the limit sets of a planar continuous dynamical system.

Theorem 2.5.1 (Poincare'-Bendixson) *Given a differentiable real dynamical system defined on an open subset of the plane, then every non-empty compact limit set of an orbit, which contains only finitely many fixed points, is either*

- *a fixed point;*
- *a limit cycle;*
- *a connected set composed of a finite number of fixed points together with homoclinic and heteroclinic orbits connecting these.*

The Poincare'-Bendixson theorem will be widely used to estabilish that stable equilibrium points of a certain system are the only limit sets of the system itself. Notice that in order to do that we need to rule out the presence of periodic solutions and a sufficient condition is provided by the *Bendixson-Dulac theorem*.

Theorem 2.5.2 (Bendixon-Dulac) *Let us consider the plane autonomous system associated to the vector field* $V(x,y) = < f(x,y), g(x,y) >$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x,y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x,y) \end{cases}$$

Let R *be a simply connected region of the plane. If the divergence of* V *has the same sign* ($\neq 0$) *almost everywhere in* R*, i.e. if*

$$\nabla \cdot V = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} > 0 \qquad \text{or} \qquad \nabla \cdot V = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} < 0 \qquad \forall\,(x,y) \in R$$

then the system has no periodic solutions lying entirely within the region R*.*

Proof *Without loss of generality, let us assume that*

$$\nabla \cdot V = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} > 0 \qquad \forall\,(x,y) \in R$$

Let C *be a closed trajectory of the plane autonomous system in* R*. Let* D *be the interior of* C*. Then by Green's Theorem we have that:*

$$\iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \oint_C (-g\,\mathrm{d}x + f\,\mathrm{d}y) = \oint_C \left(-\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}x + \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}y \right) = 0$$

This is a contradiction as we assumed a divergence greater than zero in R*, so there can be no such closed trajectory* C*.* ∎

The next result characterizes stable manifolds of dynamical systems, and will come handy estabilish some results about boundary lines of basins of attraction.

Theorem 2.5.3 (Stable-manifold) *Let $F \in C^r$ be a vector field of $\mathbb{R}^n$, where $r \geq 2,\ r \in \mathbb{N}$. Let us suppose that p is an hyperbolic fixed point of F. Then it exists a manifold $W_p \in C^r$ of dimension n, called stable manifold, such that:*

- *$p \in W_p$;*
- *Locally to p, the tangent space of W_p coincides with the stable space of the linearization of F at p;*
- *W_p is locally invariant and the restriction of the dynamical system $\dot{x} = F$ to W_p coincides with the restriction of an asintotically stable system;*
- *W_p is locally unique.*

2.6 important examples: the voter and majority models

In this part of the chapter we will consider the already presented voter and majority models, which will be analyzed by means of the theory that we have introduced in this chapter. In particular, we will consider the homogeneous plain version of such models, mainly under the mean field assumption. Afterwards, we will introduce a simple generalized voter model in presence of two subpopulations to give a foretaste of heterogeneous models.

2.6.1 *Analysis of the voter model*

We recall that in the voter model, at each discrete time step the interaction between individuals can be described as follow:

- pick a random node (the voter).
- the voter adopts the state of a random neighbor with probability q, otherwise it keeps its opinion.

Notice that the voter model is an example of dynamics induced by an homogeneous gossip interaction kernel. Indeed, the interaction kernel can be written in the form

$$\Theta(x_v, x'_v \mid \mathbf{x}) = \sum_{w \in \mathcal{N}_v} L_{vw} \theta(x_v, x'_v \mid x_w) \tag{47}$$

where L defines a simple random walk on the network and

$$\theta(x, x' \mid x_w) = \begin{cases} q & \text{if} \quad x'_v = x_w \\ 1-q & \text{if} \quad x'_v = x_v \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

In the mean field hypothesis, we have that the transition probabilities for the type process are given by $Q_{p,p'} = p_x p_{x'} q$. Notice that the anonymous interaction kernel is $\Pi(x, x' \mid p) = p_{x'} q$. If we consider a binary state space $\mathcal{X} = \{0, 1\}$, we can apply the hydrodynamic limit argument in order to find the corresponding ODE. Notice that in this case, if we call z the fraction of ones in the population, then clearly we have $q^+(z) = (1-z)zq$ and $q^-(z) = z(1-z)q$. Thus, the drift operator becomes $\mathcal{F}(z) = (1-z)zq - z(1-z)q = 0$, which means that the voter model has no drift.

Indeed, let $z(0) = z_0$ be the initial fraction of ones in the population, the Cauchy problem is readily written as:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}\tau} = 0 \\ z(0) = z_0 \end{cases} \tag{49}$$

which gives the trivial solution $z(\tau) = z_0$. Hence, in the plain voter model on the complete graph, the initial fraction of 1's is preserved in time (we will see that this holds true also in more general networks). This is apparently in contrast with the fact that, for finite N, the system will asymptotically converge to the all 0's or to the all 1's configuration. However, the fact that we do not appreciate any dynamic only means that, on avarage, nothing will happen in a time τ scaling with N as the system "oscillates" nearby the initial condition. Eventually, these stochastic fluctuations will bring the system to one of the two absorbing states (all ones or all zeros) but this phenomena will happen for larger time scales and thus they cannot be seen in the ODE.

Because of its simplicity, analytical results are available for the voter model also when the dynamic takes place on general networks; we briefly sketch how this more general analysis can be carried out. To begin with, the voter is intrinsically a linear model, as already mentioned in the first chapter. If we call $\mathbf{x}(t) \in \mathcal{X}^{\mathcal{V}}$ the detailed configuration at time $t \in \mathbb{N}$, the evolution equation for $\mathbf{x}(t)$ is of the kind

$$\mathbf{x}(t+1) = P(t) \cdot \mathbf{x}(t) \tag{50}$$

where, in case at time t the node v copies the opinion of node w, then $P(t) \in \mathcal{X}^{\mathcal{V}\times\mathcal{V}}$ is a diagonal matrix with all 1's on the diagonal but the element $(v, v) = 0$ and the element $(v, w) = 1$.

$$P(t) = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & & & & & \vdots & \vdots \\ \vdots & & \ddots & & \ddots & & \vdots & \vdots \\ \vdots & & & 1 & & & 0 & \vdots \\ \vdots & & & & 0 & & 1 & \vdots \\ \vdots & & & \ddots & & 1 & 0 & \vdots \\ \vdots & & & & & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} = \Lambda^{(v,w)} \tag{51}$$

Notice that $P(t)$ changes according to a simple probability distribution:

$$\begin{cases} \mathbb{P}(P(t) = \Lambda^{(v,w)}) = \dfrac{1}{N}\dfrac{1}{d_v}A_{v,w}\, q \\ \mathbb{P}(P(t) = I) = 1 - q \end{cases} \tag{52}$$

where A is the adjacency matrix associated to the graph $\mathcal{G}$ where the process is defined and I denotes the identity matrix. Starting from this, one can study the avarage dynamic

$$\overline{\mathbf{x}}(t+1) = \overline{P} \cdot \overline{\mathbf{x}}(t) \tag{53}$$

where $\overline{\mathbf{x}}(t) = \mathbb{E}[\mathbf{x}(t)]$ and $\overline{P} = \mathbb{E}[P(t)]$. The above equation describes a consensus dynamic driven by the stochastic matrix $\overline{P}$, which turns out to describe a lazy random walk on the graph $\mathcal{G}$. If $\mathcal{G}$ is strongly connected, one can use the theory of consensus dynamics to prove that the voter model always converges to an absorbing state which is a consensus point, that is, all the individuals share the same opinion; these states are also called "pure configurations".

Closed results regarding exit probability and exit time can also be obtained in general cases. For instance, if we consider a binary state space $\mathcal{X} = \{0, 1\}$ (notice that in this case the pure configurations are $0\mathbb{1}$ and $\mathbb{1}$) and a connected regular graph, we can notice the following striking fact regarding the voter dynamics: the probability of a node making a transition from 0 to become 1 is always equal to that of a node making a transition from 1 to 0. If we denote by $N(t)$ the process describing the number of 1's in the population we thus have that $\mathbb{E}[N(t+1)] = \mathbb{E}[N(t)]$ for all instants t; this implies that $\mathbb{E}[N(t)] = N(0)$, which in turn means that the avarage number of 1's (or 0's) is a conserved quantity. In particular, denoting by $N(\infty)$ the asymptotic number of 1's, we can say that

$$\mathbb{P}(\mathbf{x}(\infty) = \mathbb{1}) = \mathbb{P}(N(\infty) = N) = \frac{\mathbb{E}[N(\infty)]}{N} = \frac{\mathbb{E}[N(0)]}{N} = \frac{N(0)}{N} \tag{54}$$

This result states that the exit probability related to the absorbing state $\mathbb{1}$ is equal to the initial fraction of ones in the population. Notice that this is true independently on how opinions are located in the network.

Computing exactly the exit time is instead more tricky as it does depend on the network topology. Several closed results are available for general networks as one can see in [bibl voter time]

2.6.2 *Analysis of the majority model*

Differently from the voter, the majority dynamics consider multibody interactions and as result the model is in general non linear. Because of this, the study of the majority model on general networks can be extremely difficult and analytical results are known under mean filed-like assumptions. Notice that, in general graphs, we can say for sure that pure configurations are absorbing state for the underlying Markov chain, but in general, depending on the graph topology, there will be many more. It can be proven that on the complete graph the pure configurations are the only absorbing states.

The associated interaction kernel can be written as follows:

$$\Theta_v(x, x' \mid \mathbf{x}) = \begin{cases} \dfrac{1}{|\mathbf{x}^{\max}|} & \text{if} \quad x' \in \mathbf{x}^{\max} \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

where $\mathbf{x} \in \mathcal{N}_v$ and

$$\mathbf{x}^{\max} = \arg\max_{x \in \mathcal{X}} |\{w \in \mathcal{N}_v \mid x_w = x\}| \tag{56}$$

Notice that the interaction kernel (55) is not a gossip interaction kernel.

Under the mean field assumption, one can write the transition probability of the associated type process:

$$Q_{p,p'} = \begin{cases} p_x \dfrac{1}{|\mathbf{y}^{\max}|} & \text{if} \quad x' \in \mathbf{y}^{\max} \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

where in this case $\mathbf{y}^{\max} = \arg\max(p)$.

We analyze now more in detail the case when $\mathcal{X} = \{0, 1\}$ and under the mean field assumption. In particular, we consider a k-majority model, whose interaction dynamics can be described as follows:

- pick a random node.
- the chosen node looks at k of its neighours (we assume k odd).
- the chosen node copies the majority opinion of this set of people.

In order to construct the transition probabilities, we need the probability to have a majority of a certain opinion, say 1, in a set of k people chosen over N people, given the fraction of ones in the population z at a certain time instant. Then, one can prove that such a probability is given by

$$M_k^N(z) = \sum_{i=\lceil \frac{k}{2} \rceil}^{k} \frac{\binom{zN}{i}\binom{(1-z)N}{k-i}}{\binom{N}{k}} \tag{58}$$

If we consider the hydrodynamic limit, we obtain

$$\lim_{N\to\infty} \sum_{i=\lceil \frac{k}{2} \rceil}^{k} \frac{\binom{zN}{i}\binom{(1-z)N}{k-i}}{\binom{N}{k}} = \sum_{i=\lceil \frac{k}{2} \rceil}^{k} \binom{k}{i} z^i (1-z)^{k-i} = M_k(z) \tag{59}$$

Thus, we have that $q^+(z) = (1-z)M_k(z)$ and $q^-(z) = zM_k(1-z)$. The simplest case is when $k = 3$, which will be considered also in the heterogeneous models that we are going to analyze in the next chapter. For this value of k, one can evaluate $M_3(z)$ and some algebra shows that the ODE (38) becomes

$$\frac{dz}{d\tau} = z(1-z)(2z-1) \tag{60}$$

Although the solutions of the above equation can be found even in closed form, a quick study of the right-hand side immediately shows that there exist three equilibria: $z_1 = 0$, $z_2 = 1$, which are stable and $z_3 = \frac{1}{2}$, which instead is unstable and separates the two basins of attraction of the stable equilibria. This means that if we start with more ones than zeros, then the system will converge to the absorbing state where all individuals have opinion 1; the contrary happens if we start with more zeros than ones. The following figure 5 shows same solutions of ODE (60) for different initial conditions.

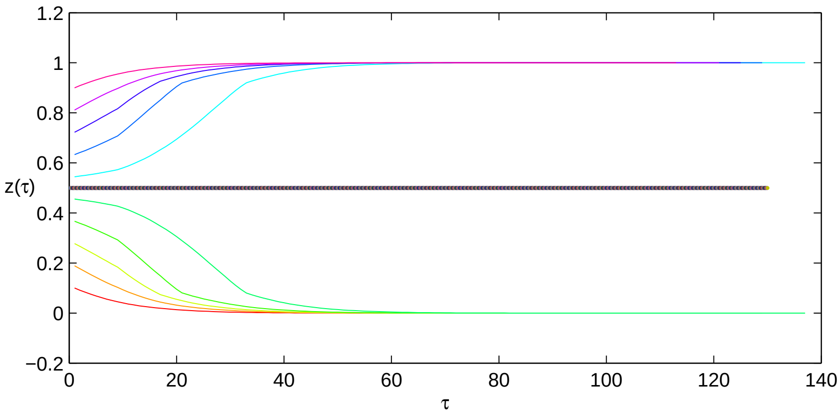


Figure 5: Family of solutions of ODE (60) simulated in MATLAB.

Notice that in this case the asymptotics are perfectly in agreement with the behavior of $z(t)$ for finite N. Differently from the voter model, in this case, $z(t)$ gets close to the absorbing states in a time which scales with N and thus the dynamic is captured by the ODE.

Part II

DESIGN OF HETEROGENEOUS MODELS

3

NODE-BASED HETEROGENEOUS MODELS

In this chapter we are going to model different heterogeneous dynamics using voter and majority models as building blocks. We will consider a population splitted into two heterogeneous subpopulations, namely: $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$. The analysis will be mainly carried out under the mean field assumption and considering a binary state space, which means that each individual is endowed with two possible opinions, namely $\mathcal{X} = \{0, 1\}$. Moreover, in the following we will denote with x and y the fraction of ones in the two subpopulations respectively. Thus, the mathematical framework is the one described in section 2.4.3.

The models that we are going to analyze are characterized by individuals who behave in different ways independently on who they are interacting with. For instance, and invidial belonging to subpopulation $\mathcal{V}_2$ will follow a different dynamic with respect to individuals belonging to $\mathcal{V}_1$ independently on whether it interacts with people belonging to its subpopulation or not. We may also say that the heterogeneity affects the nodes (the individuals) and not the edges (the relationships) of the network.

3.1 HETEROGENEOUS VOTER MODEL

The model that we are going to build aims to generalize the voter model by considering the interaction between populations characterized by different levels of open-mindedness. In particular, individuals belonging to $\mathcal{V}_1$ are more open mind than those in $\mathcal{V}_2$ and they are more likely to change their opinion when interacting with other people. This heterogeneity is modelled by considering two different copying probabilities for the two population, such that $q_1 > q_2$.

The interaction dynamic is defined by considering the following interaction kernel:

$$\theta_{v,w}(x, x'|y) = \begin{cases} q_1 & x' = y \\ 1 - q_1 & x' = x \\ 0 & \text{otherwise} \end{cases} \qquad v \in \mathcal{V}_1 \tag{61}$$

$$\theta_{v,w}(x, x'|y) = \begin{cases} q_2 & x' = y \\ 1 - q_2 & x' = x \\ 0 & \text{otherwise} \end{cases} \qquad v \in \mathcal{V}_2 \tag{62}$$

Let us find the system of differential equation describing this model in the mean-field approximation. In order to do that, we need the probabilities of increase and decrease the number on ones in each population.

We have:

$$\begin{aligned} q_1^+(x, y) &= \mathbb{P}(\text{choose an agent in state } 0 \text{ in } \mathcal{V}) \\ &\cdot \mathbb{P}(\text{choose any agent in state one to interact with}) \cdot \mathbb{P}(\text{copy the state}) \\ &= (1 - x)(\rho_1 x + \rho_2 y) q_1 \end{aligned}$$

The other probabilities evaluate in the same way. For variable x, we obtain the following equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1 - x)(\rho_1 x + \rho_2 y) q_1 - x(1 - \rho_1 x - \rho_2 y) q_1 \tag{63}$$

We can do exaclty the same for variable y. After simplifying the equations, we end up with the following planar system of ODE:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = q_1 \rho_2 (y - x) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = q_2 \rho_1 (x - y) \end{cases} \tag{64}$$

This is a linear system and the solution may be found even in closed form. However, we can say everything without actually solving the system, starting with the stability analysis.

The jacobian matrix is given by:

$$J = \begin{bmatrix} -q_1 \rho_2 & q_1 \rho_2 \\ q_2 \rho_1 & -q_2 \rho_1 \end{bmatrix} \tag{65}$$

and we immediately see that:

$$\text{Tr}(J) = -q_1 \rho_2 - q_2 \rho_1 < 0 \qquad \text{Det}(J) = q_1 q_2 \rho_1 \rho_2 - q_1 q_2 \rho_1 \rho_2 = 0 \tag{66}$$

and this implies that one eigenvalue is 0 and the other one is negative. In this simple case this means that the system is charachterized by simple stability. Indeed, the system has infinitely many equilibria, which can be found in the kernel of J:

$$J\begin{bmatrix}x\\y\end{bmatrix}=0 \qquad \Longleftrightarrow \qquad \begin{cases}q_1\rho_2(y-x)=0\\q_2\rho_1(x-y)=0\end{cases} \qquad \Longleftrightarrow \qquad x=y$$

This means that the equilibria lie on the main diagonal of the domain.

Of course, which of these equilibria will be reached by the system depends on the initial conditions, i.e. the fractions of 1's in the two populations at time 0. To find the explicit dependence, we can observe that in this case we can even find the explicit equation of the trajectories. Indeed, by dividing the second equation by the first one in (64), we get:

$$\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{q_2\rho_1}{q_1\rho_2} \qquad \Longleftrightarrow \qquad y=-\frac{q_2\rho_1}{q_1\rho_2}x+k \tag{67}$$

where $k\in\mathbb{R}$ is a constant. We see that the trajectories are parallel lines.

We can find the equation of a particular trajectory by imposing the passage through the generic initial point of a trajectory (x_0,y_0), we readily find:

$$q_2\rho_1x+q_1\rho_2y=q_2\rho_1x_0+q_1\rho_2y_0 \tag{68}$$

We notice that the quantity $E=q_2\rho_1x+q_1\rho_2y$ is conserved and it is a first integral for the system. Similarly to the plain homogeneous voter, even in the heterogeneous case we thus have a conservation law but, rather than the total fraction of ones of the homogeneous voter model, the conserved quantity this time is a weighted avarage of the two fractions of ones, where the weights are the two copying probabilities.

In order to find which one of the infinitely many equilibria will be reached depending on the initial conditions, we just impose the intersection of the above trajectory with the equilibria line $y=x$. Trivially we get:

$$(q_2\rho_1+q_1\rho_2)x_{eq}=q_2\rho_1x_0+q_1\rho_2y_0 \qquad \Longleftrightarrow \qquad x_{eq}=y_{eq}=\frac{q_2\rho_1x_0+q_1\rho_2y_0}{q_2\rho_1+q_1\rho_2} \tag{69}$$

Thus, at equilibrium we have a total fraction of 1's given by:

$$\frac{n_{eq}}{N}=\rho_1x_{eq}+\rho_2y_{eq}=\frac{q_2x_0+q_1y_0}{q_2\rho_1+q_1\rho_2} \tag{70}$$

Similarly to the classic voter model, even in this case we have two absorbing states for the original model: all ones or all zeros (consensus) but in this case the mean-field approximation predicts a magnetization of ones density, while in the plain voter model we don't appreciate any dynamic. In the following picture we show the phase portrait of the system.

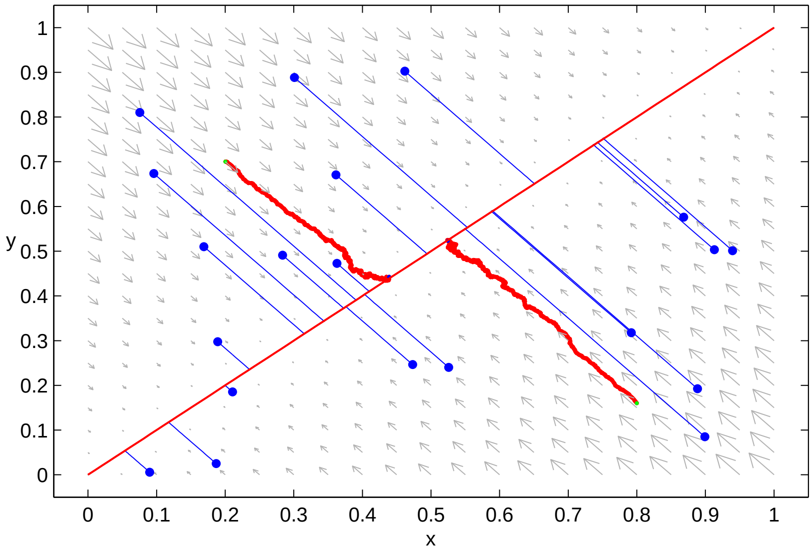


Figure 6: Phase portrait of the heterogeneous voter model. The equilibria lie on the red line. In red, two sample paths of the Markov chain are shown.

3.2 VOTER-ANTI VOTER MODEL

The first heterogeneous model that we are going to consider involves the voter dynamics and explores the effect of a subpopulation $\mathcal{V}_2$ that behaves in an anti social fashion. In this case infact, while population $\mathcal{V}_1$ follows a plain voter model, population $\mathcal{V}_2$ behaves in an opposite way, in an anti-voter fashion. This means that a generic individual belonging to such population, instead of copying a random neighbour's opinion with a certain probability, will copy the *opposite* opinion of the same neighbour.

This kind of dynamic can be defined by considering the following interacting kernel:

$$\theta_{v,w}(x,x'|y) = \begin{cases} q_1 & x' = y \\ 1-q_1 & x' = x \\ 0 & \text{otherwise} \end{cases} \qquad v \in \mathcal{V}_1 \tag{71}$$

$$\theta_{v,w}(x,x'|y) = \begin{cases} q_2 & x' = \bar{y} \\ 1-q_2 & x' = x \\ 0 & \text{otherwise} \end{cases} \qquad v \in \mathcal{V}_2 \tag{72}$$

where $\bar{y}$ denotes the opposite state with respect to y. Notice that this is well defined only in the case of a binary state space.

Let us find the differential system describing the dynamics of the model when considering the hydrodynamic limit.

We may notice the equation governing the dynamic of pupulation $\mathcal{V}_1$ involving variable x, following a plain voter model, can be derived in the same way as shown in the analysis of the heterogeneous voter model that we have seen at the end of chapter 2; in particular, the equation will be the same as equation (63). For what concerns population $\mathcal{V}_2$, we just need to invert the probabilities to choose any individual in state zero or one. More in details we have that

$$q_2^+(x,y) = (1-y)(1-\rho_1 x - \rho_2 y)q_2$$

$$q_2^-(x,y) = y(\rho_1 x + \rho_2 y)q_2$$

which brings to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (1-y)(1-\rho_1 x - \rho_2 y)q_2 - y(\rho_1 x + \rho_2 y)q_2 \tag{73}$$

After simplifying the above equation, the dynamical system of the model is the following:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = q_1\rho_2(y-x) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = q_2(1-(1-\rho_2)x-(1+\rho_2)y) \end{cases} \tag{74}$$

This is a linear system that may be solved even exactly. However, what we are really interested in is the asymptotic behaviour, which can be analyzed by performing the stability analysis.

Let us find the equilibria of the above system. It is immediate to see that $\dot{x} = 0 \iff x = y$. Then, by substituting in the second equation we obtain:

$$q_2(1-(1-\rho_2)x-(1+\rho_2)x) = 0 \quad \iff \quad x = \frac{1}{2} \tag{75}$$

Hence, the only equilibrium of the system is given by: $(x_{eq},\, y_{eq}) = (\frac{1}{2}, \frac{1}{2})$. Notice that this point is the state of maximum entropy of the system, where in the two populations coexist the two opinions with same weights.

To check whether the equilibrium it is stable or unstable, we consider the jacobian matrix of the system evaluated in such a point:

$$J = \begin{bmatrix} -q_1\rho_2 & q_1\rho_2 \\ -q_2(1-\rho_2) & -q_2(1+\rho_2) \end{bmatrix} \tag{76}$$

from which we see that:

$$\text{Tr}(J) = -q_1\rho_2 - q_2(1+\rho_2) < 0 \qquad \text{Det}(J) = q_1q_2(1+\rho_2)\rho_2 + q_1q_2(1-\rho_2)\rho_2 > 0 \tag{77}$$

This implies that the jacobian J has two negative eigenvalues, which in turn implies the global asymptotic stability of the equilibrium point independently on the value of ρ_2. The following figure shows a phase portrait of the voter-antivoter model.

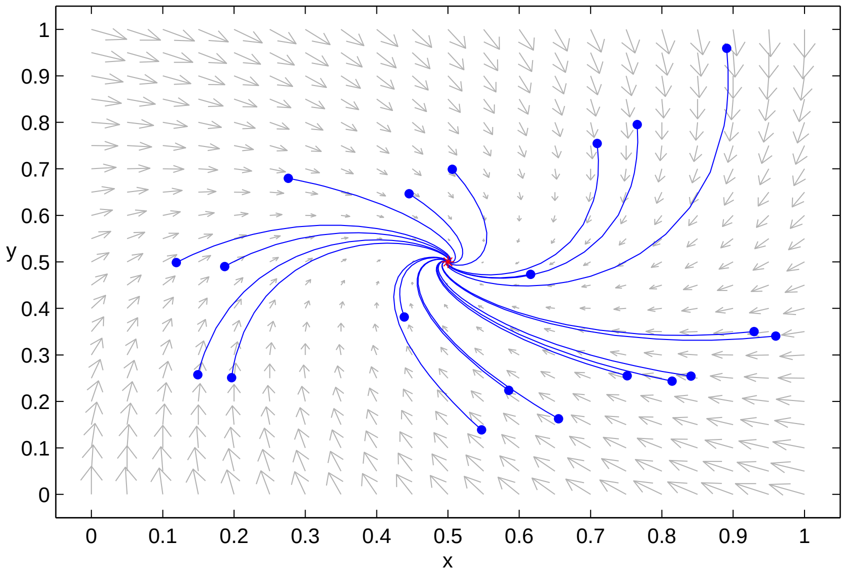


Figure 7: Phase portrait of the voter-antivoter model simulated in MATLAB for $q_1 = 0.7$, $q_2 = 0.3$ and $\rho_2 = 0.35$. In red, the stable equilibrium point.

Voter-anti voter model dynamics

The analysis that we have carried out shows that the anti-voter individuals bring the system to its maximum disorder and this happens *independently* on their number and the copying probabilities of both populations. This is in complete contrast with the usual consensus dynamic that characterized the voter model and implies that a perturbation caused even by a small group of anti social voter is enough to bring the system to an highly disordered state.

3.3 MAJORITY MODEL WITH ANTI-MAJORITY INDIVIDUALS

The model that we are going to consider in this section aims to describe an anti social behaviour involving the majority dynamic. In particular, individuals belonging to population $\mathcal{V}_1$ behaves according to a 3-majority model, while people belonging to $\mathcal{V}_2$ follows a 3-*minority dynamic*, that is, an individual will copy the minority opinion among three randomly chosen people.

In order to build the differential system governing the model in the hydrodynamic limit, we need the probability to increase and decrease the fraction of ones for both populations. For what concerns population $\mathcal{V}_1$, one can use equation (59) to write down the probability to find a majority of ones among three randomly chosen individuals; let $z = \rho_1 x + \rho_2 y$ be the total fraction of ones in the whole population, then:

$$M_3(z) = \sum_{i=2}^{3} \binom{3}{i} z^i (1-z)^{3-i} = 3z^2 - 2z^3 \tag{78}$$

Notice that the probability to have a minority of ones is obviously equal to the probability of having a majority of zeros, which is given by

$$M_3(1-z) = \sum_{i=2}^{3} \binom{3}{i} (1-z)^i z^{3-i} = 2z^3 - 3z^2 + 1 \tag{79}$$

Thus, we have that $q_1^+(x,y) = (1-x)M_3(z)$ and $q_1^-(x,y) = xM_3(1-z)$ are the transition probabilities for population $\mathcal{V}_1$. Instead, for individuals belonging to $\mathcal{V}_2$ we have $q_1^+(x,y) = (1-y)M_3(1-z)$ and $q_1^-(x,y) = yM_3(z)$. The differential system governing the model is then readily written as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (1-x)M_3(z) - xM_3(1-z) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (1-y)M_3(1-z) - yM_3(z) \end{cases} \tag{80}$$

This can be also be written explicitely in terms of variables x and y:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2\rho_1^3 x^3 + 3\rho_1^2(2\rho_2 y + 1)x^2 + (6\rho_1(1-\rho_2 y)\rho_2 y - 1)x + \rho_2^2 y^2(3 - 2\rho_2 y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 - y - (3 - 2\rho_1 x - 2\rho_2 y)(x - \rho_2 x + \rho_2 y)^2 \end{cases} \tag{81}$$

Differently from the heterogeneous voter models, this system is clearly non linear and looking for an exact solution seems hopelessly complicated. However, what really matters, from the point of view of the mean-field approximation, is the asymptotic behaviour of the solutions.

Stability analysis

In order to find the equilibria we must solve the non linear algebrical system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 0 \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 0 \end{cases}$$

This non-linear system admits three solutions and with some effort we can show that they are given by:

$$(x_{eq},\, y_{eq})_{\mathrm{I}} = \left(\frac{1}{2},\, \frac{1}{2}\right)$$

$$(x_{eq},\, y_{eq})_{\mathrm{II}} = \left(\frac{1}{2} - \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}},\, \frac{1}{2} + \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}}\right)$$

$$(x_{eq},\, y_{eq})_{\mathrm{III}} = \left(\frac{1}{2} + \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}},\, \frac{1}{2} - \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}}\right)$$

We can see that equilibrium I always exists while II and III are reals only for $\rho_2 < 1/6$ and $\rho_2 > 1/2$. Moreover, a quick study shows that for $\rho_2 > 1/2$ equilibria II and III are not admissible as some component is negative. Finally, we notice that equilibria II and III are symmetric to each other with respect to the main diagonal of the phase space (equilibrium I is symmetric to itself as it lies on the main diagonal). This is an obvious consequence of the fact that our system is perfectly symmetric with respect to fractions of ones and zeros.

From this result we can already guess that some bifurcation occurs around $\rho_2 = \frac{1}{6}$. Let us carry out the stability analysis to check that.

Equilibrium I

Let us start with equilibrium I. The jacobian matrix of the system evaluated in that point is:

$$J_I = \begin{bmatrix} \frac{1}{2}(1 - 3\rho_2) & \frac{3}{2}\rho_2 \\ \frac{3}{2}(1 - \rho_2) & -\frac{3}{2}\rho_2 - 1 \end{bmatrix} \tag{82}$$

The eigenvalues are:

$$\lambda_1 = -1 \qquad \lambda_2 = \frac{1}{2}(1 - 6\rho_2)$$

from which we immediately see that the equilibrium is asymptoticaly stable when $\rho_2 > 1/6$ and is unstable for $\rho_2 < 1/6$. Notice that this equilibrium corresponds to the state of maximum entropy for the system.

Equilibrium II
The Jacobian matrix is:

$$J_{II} = \begin{bmatrix} \dfrac{6\rho_2^2 - 8\rho_2 + 1}{2\rho_2 - 1} & \dfrac{6\rho_2^2}{2\rho_2 - 1} \\ \dfrac{6(1-\rho_2)\rho_2}{2\rho_2 - 1} & 3\rho_2 + \dfrac{\rho_2 + 1}{2\rho_2 - 1} \end{bmatrix} \tag{83}$$

The eigenvalues are:

$$\lambda_1 = -1 \qquad\qquad \lambda_2 = 6\rho_2 - 1$$

and this time the equilibrium is asymptoticaly stable when $\rho < \frac{1}{6}$ and unstable for $\rho_2 > \frac{1}{6}$, as we could expect.

In this case we have that the total fraction of ones is given by:

$$\rho_1 x_{eq} + \rho_2 y_{eq}|_{\text{II}} = \frac{1}{2} - \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{2\rho_2 - 1}} \tag{84}$$

Equilibrium III
The stability resulst are exactly the same as equilibrium II.
In this case we have a fraction of ones given by:

$$\rho_1 x_{eq} + \rho_2 y_{eq}|_{\text{III}} = \frac{1}{2} + \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{2\rho_2 - 1}} \tag{85}$$

The results that we have just derived show that a supercritic pitchfork bifurcation occurs around $\rho_2 = \frac{1}{6}$.

Now that we have discussed the local equilibrium properties, in the following we use the theory of planar differential systems in order to the dynamics of the model from the global point of view, in both cases for ρ_2.

Proposition 3.3.1 *The equilibrium I is globally stable for the majority-minority model for $\rho_2 > 1/6$. Moreover, the three equilibria I, II and III are the only attractors for the system.*

Proof *Since we are dealing with planar continuous dynamic systems, the Poincare'-Bendixon theorem comes in help. Indeed, in our case it's enough to exclude the presence of attractive limit cycles to conclude that our equlibrium points are the only attractors for the* *system. In particular, this would mean that equilibrium I is globally stable (for $\rho_2 < 1/6$), as its attractive region would be the whole space.*

In order to rule out the presence of limit cycles, we use the Bendixson-Dulac theorem. By computing the divergence of the vector field we have that:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 6 \underbrace{(2\rho_2 - 1)}_{>0 \text{ or } <0} \underbrace{(-\rho_1 x - \rho_2 y)}_{<0} \underbrace{(1 - \rho_1 x - \rho_2 y)}_{>0} -2 \tag{86}$$

We notice that when $\rho_2 \geq 1/2$, then $2\rho_2 - 1 \geq 0$ and the divergence is negative. When instead $\rho_2 < 1/2$ the product of the three factors in (27) is positive and we can say anything just looking at the signs. However, we can notice that in this case the maximum value for the divergence is achieved for $x = y = 1/2$ and this value is $-(3\rho_2 + 1/2)$, which is negative as well and this implies that the divergence is negative even for $\rho_2 < 1/2$. This holds $\forall\, (x, y) \in [0, 1]^2$. Thus, the divergence must be always negative over the domain in any case. By Bendixson-Dulac theorem, no limit cycles can exist in the domain, which in turn implies that equilibrium I is globally stable because of Poincare'-Bendixson theorem. ∎

Figure 8 shows a phase portrait of the system when $\rho_2 > \frac{1}{6}$.

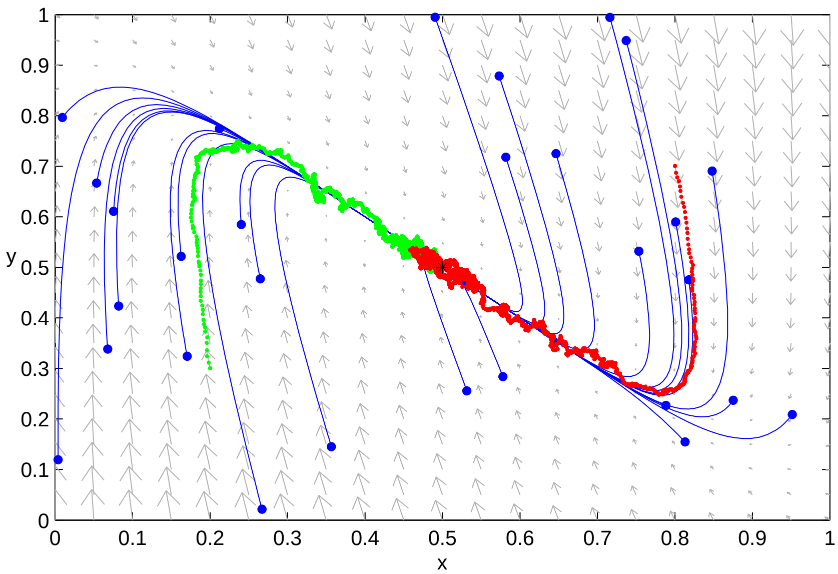


Figure 8: Phase portrait of the model in the case $\rho_2 > 1/6$ simulated in MATLAB. We can see that equilibrium I is globally stable for the system. In red and green two sample paths of the underlying Markov process are shown.

We can say almost everything even about the two basins of attraction in the case $0 < \rho_2 < \frac{1}{6}$, when two stable equilibria coexist. Figure 9 shows the phase portrait of the model in the case $\rho_2 = 0.08$. The black stars denote the two stable equilibria, and the trajectories have different colours depending on which equilibrium are converging to. We can clearly appreciate the two basins of attraction as well as their separation boundary, which is the black-thick line in the picture. In this case infact, the boundary turns out to be the stable manifold of the hyperbolic equilibrium I (which is an unstable saddle).

Indeed, since we are dealing with a smooth vector field and equilibrium I is hyperbolic, by the Stable-Manifold theorem we know that the unique stable manifold relative to equilibrium I is a smooth manifold and its tangent space has the same dimension as the stable manifold of the linearization of the system at equilibrium I. This means that we can at least approximate the stable manifold (our boundary) with the stable space of the linearized system around equilibrium I. Such stable space is the eigenspace associated with the negative eigenvalues of jacobian (82). Of course, since we have only one negative eigenvalue, the stable space has dimension 1, which means that is a line.

By computing the eigenvectors of the jacobian matrix (82), it's easy to see that the stable space is given by $L_I = \{v \in \mathbb{R}^2 : v = a(\frac{\rho_2}{\rho_2 - 1}, 1)^T,\ a \in \mathbb{R}\}$. This line is our approximate boundary. Infact, simply translating it to let it pass through equilibrium I in our coordinate system, we obtain the explicit equation of the boundary line (the black one in the picture):

$$y = -\frac{1-\rho_2}{\rho_2}x + \frac{1}{2\rho_2} \quad \Longrightarrow \quad \rho_1 x + \rho_2 y = \frac{1}{2} \tag{87}$$

The above equation shows that the boundary line is the locus where the total number of ones is $\frac{1}{2}$ and the system lies in a disordered state. As it is clear from the picture, simulations confirm that this is a very good approximation of the basins of attraction's boundary, even when we are not in the neighbourhood of equilibrium I.

Given two initial fractions of 1s x_0 and y_0 in the respective populations, we can thus say the following:

- If $\rho_1 x_0 + \rho_2 y_0 > \frac{1}{2}$, so that there are more ones than zeros, then the trajectory will converge to equilibrium III.
- If $\rho_1 x_0 + \rho_2 y_0 < \frac{1}{2}$, so that there are more zeros than ones, then the trajectory will converge to equilibrium II.

This agrees with what we could have expected: if we start with more ones than zeros, the system will end up in the equilibrium with more ones than zeros (equilibrium III), viceversa it will converge to equilibrium II.

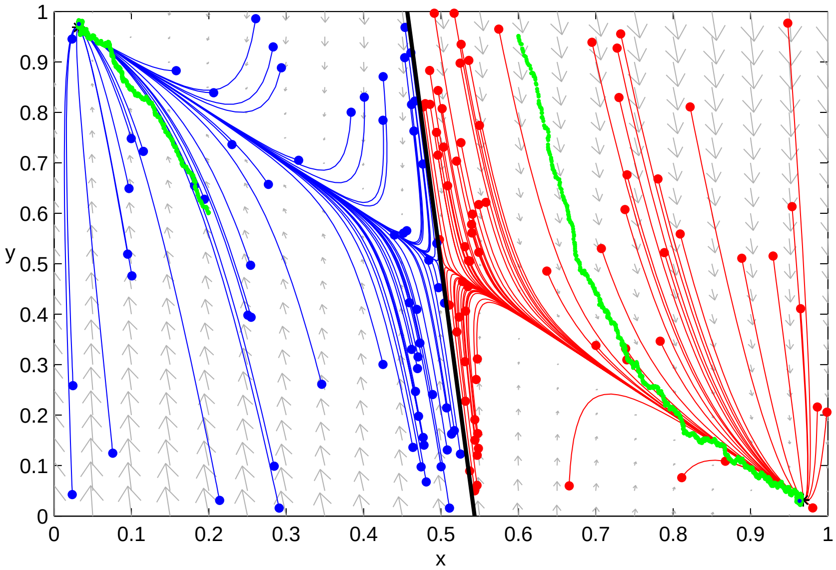
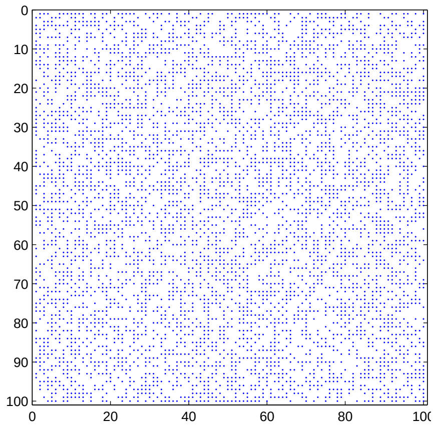
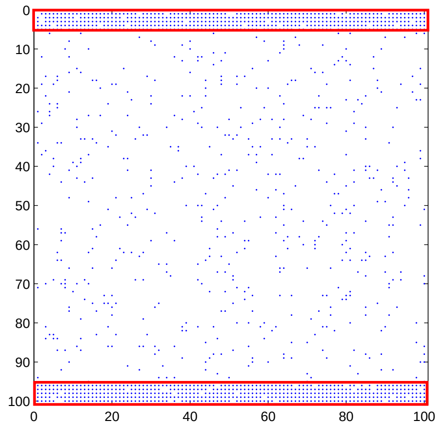


Figure 9: Phase portrait of the model in the case of $\rho_2 < 1/6$ simulated in MATLAB. In particular, we have chosen $\rho_2 = 0.08$. We can appreciate the boundary line of the basins of attraction (the black one). In green, two sample paths of the underlying Markov chain are shown.

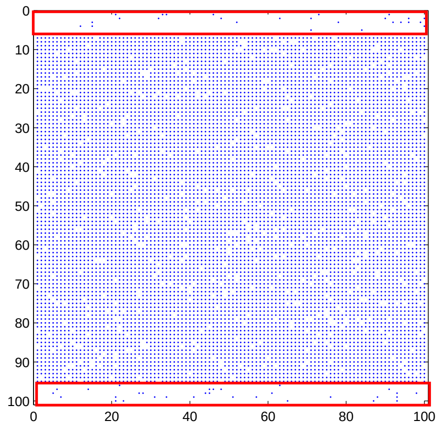
Simulations allow us to see also the polarization phenomena occurring for $\rho_2 < 1/6$. We can notice that, in both equlibria, one of the two pupulation reaches a large fraction of ones, while the other one, because of simmetry reasons, the same fraction of zeros. We can understand this also looking at the analytic expressions of equilibria II and III. This means that a state very close to consensus is reached in the two populations. This is more effectively shown in figure 10.



(a) Equilibrium I



(b) Equilibrium II



(c) Equilibrium III

Figure 10: Phase transitions of the model visualized in a sparse matrix fashion. Each dot insiede the square represents one individual and can be colored in blu or white whether its opinion is one or zero respectively. The dots inside the red rectangles are the anti majority inividuals.

The dynamics of the mixed majority-minority model

The result that we have derived show that a phase transition occurs in the model at $\rho_2 = \frac{1}{6}$. In particular, the effect of the anti majority individuals becomes really effective when $\rho_2 > \frac{1}{6}$. Indeed, above this threshold they are in enough number to bring the system to complete disorder. Instead, when $\rho_2 < \frac{1}{6}$, two stable equilibria very close to polarized consensus points appear. We have seen that the basins of attraction of these equilibria are separated by the locus of points where the total fraction of one and zero opinions are the same. If the system starts with more ones than zeros, then the system converges towards the stable point characterized by the largest fraction of ones; conversely, the other equilibrium is reached when the at the initial conditions there are more zero-opinions than one-opinions. Moreover, when $\rho_2 < \frac{1}{6}$, it is interesting to observe that the anti majority dynamic pursued by individuals belonging to $\mathcal{V}_2$ gives rise to a quasi-local consensus within the population itself. This high-level feature arising from the model is rather surprising as the anti majority dynamic tries to create disagreement and not consensus.

3.4 MAJORITY-VOTER MODEL

In this part of the chapter we will consider a mixed model, where population $\mathcal{V}_1$ follows a 3-majority model while population $\mathcal{V}_2$ a voter model. To make the two models more comparable, we introduce a copying probability even for the majority model, in the sense that an agent will copy the local majority state with a certain probability q_1, while it will keep its opinion otherwise.

Differently from the models with anti social individuals that we have seen so far (the anti-voter and the anti-majority), in this case we don't have an antisocial behaviour and both dynamics, separately considered, tend to a consensus point. In fact, we can observe that the two consensus states (all ones and all zeros) are both absorbing states for the original Markov chain of this mixed model.

Notice that we have already derived in the previous models the two equations that we need in order to define the differential system of ODE. Thus, the The mean-field dynamic is governed by the following dynamic system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2\rho_1^3x^3 - 3\rho_1^2(2\rho_2 y + 1)x^2 + (6\rho_1(1-\rho_2 y)\rho_2 y - 1)x + \rho_2^2 y^2(3 - 2\rho_2 y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = q_2(1-\rho_2)(x-y) \end{cases} \tag{88}$$

Let us carry out the stability analysis.

Stability analysis

From the second equation of the system we easily get that at equilibrium must be: $q_2(1-\rho_2)(x-y) = 0 \iff x = y$. Substituting in the first equation we obtain:

$$-2\rho_1^3x^3 + 3\rho_1^2(2\rho_2 x + 1)x^2 + (6\rho_1(1-\rho_2 x)\rho_2 x - 1)x + \rho_2^2 x^2(3 - 2\rho_2 x) = 0$$

This equation can be simplified and is equivalent to:

$$x((3-2x)x-1)=0 \tag{89}$$

It is immediate to see that the above equation admits the solutions $x_1 = 0, x_2 = \frac{1}{2}$ and $x_3 = 1$. Thus, the system admits the following three equilbria:

$$(x_{eq},\, y_{eq})_{\text{I}} = \;(0,0) \qquad (x_{eq},\, y_{eq})_{\text{II}} = \left(\frac{1}{2},\,\frac{1}{2}\right) \qquad (x_{eq},\, y_{eq})_{\text{III}} = \;(1,1) \tag{90}$$

Notice that points I and II represent the consensus states while equilibrium II is the state of maximum entropy. Let us now check the stability properties of these points.

Equilibrium I

The jacobian matrix is:

$$J_I = \begin{bmatrix} -q_1 & 0 \\ q_2(1-\rho_2) & -q_2(1-\rho_2) \end{bmatrix} \tag{91}$$

and we see that:

$$\text{Tr}(J) = -q_1 - q_2(1-\rho_2) < 0 \qquad \text{Det}(J) = q_1 q_2 (1-\rho_2) > 0$$

This implies that the jacobian has two negative eigenvalues and so equilibrium I is asimptotically stable.

Equilibrium II

The jacobian matrix is:

$$J_{II} = \begin{bmatrix} \frac{1}{2}(q_1 - 3q_1\rho_2) & \frac{3}{2}q_1\rho_2 \\ & \\ q_2(1-\rho_2) & -q_2(1-\rho_2) \end{bmatrix} \tag{92}$$

We can notice that:

$$\text{Det}(J) = -\frac{1}{2} q_1 q_2 (1-\rho_2) < 0$$

and this means that the product of the eigenvalues is negative, so one eigenvalue must be negative and the other one positive. Hence, we conclude that the equilibrium point is an unstable saddle point.

Equilibrium III

The jacobian is exactly the same as equilibrium I (91). Hence, equilibrium III is asimptotically stable as well.

From the stability analysis we can observe that the mean-field dynamic matches the asymptotic dynamic of the original markov chain, as the stable equilibria are the consensus points, which in turn are exactly the absorbing states of the underlying stochastic Markov process. Moreover, those equilibria are the only attractors for the system, as we prove in the following.

Proposition 3.4.1 *The three equilibria I, II and III are the only limit sets for the system.*

Proof *By Poincare'-Bendixson theorem, we just need to rule out the presence of limit cycles and in order to do that we use the Bendixson-Dulac theorem.*

Computing the divergence of the vector field we can see that :

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \underbrace{-q_2\rho_1}_{<0} + q_1(-1 \underbrace{-6\rho_1}_{<0} \underbrace{(-\rho_1 x - \rho_2 y)}_{<0} \underbrace{(1 - \rho_1 x - \rho_2 y)}_{>0}) \tag{93}$$

Notice that we can say anything just looking at the signs as in the second addendum on the right hand side we have the sum between a positive and a negative term. However, we the help of Mathematica one can see that the divergence doesn not change its sign for any values of the parameters.

By Bendixson-Dulac theorem, no limit cycles can exist in the domain, which in turn implies that equilibria I, II and III are the only attractors for the system because of Poincare'-Bendixson theorem. ∎

Even in this case, we can say almost everything about the two basins of attraction. Indeed, we can use the same argument that we used in the majority-anti-majority model to approximate the boundary of the basins with the stable space of the saddle equilibrium point. By computing the eigenvector associated to the negative eigenvalue of jacobian (92), we find that the corresponding eigenspace (the stable space) is describes a line whose angular coefficient is given by:

$$m = \frac{4q_2\rho_1}{2q_2\rho_1 - q_1(3\rho_2 - 1) - \sqrt{8q_1q_2\rho_1 + (q_1 - 2q_2\rho_1 - 3q_1\rho_2)^2}} \tag{94}$$

Translating it to let it pass through equilibrium II in our coordinate system we obtain the equation of the boundary line: $y = mx + \frac{1}{2}(1 - m)$. Notice that this time the boundary does not have a simple interpretation as a particular locus as in the majority-anti-majority model. Neverthless, simulations perfectly agree with this result, as is shown in figure 11 where we plot the phase portrait of the model and the boundary line (the black-thick line).

It is interesting to investigate the extreme cases that can occur in the model depending on the population fractions. We can notice that, keeping q_1 and q_2 fixed, when we have no voter individuals (all majority's), i.e. $\rho_2 \to 0$, then $m \to -\infty$

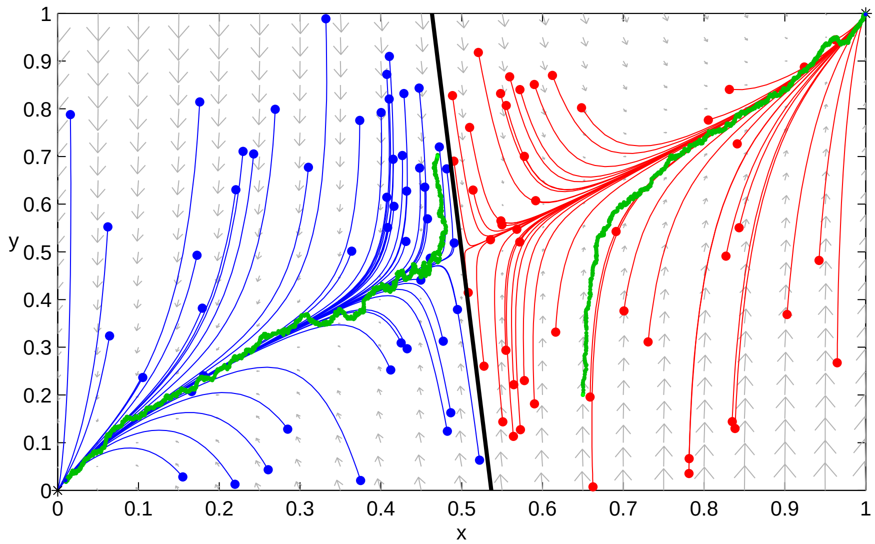


Figure 11: Phase portrait of the model with $\rho_2 = 0.1$, $q_1 = 0.4$ and $q_2 = 0.7$ simulated in MATLAB. We plot also two sample paths of the underlying Markov chain (in green), which considers a population of $N = 10000$ individuals.

and the boundary line tends to be vertical. Conversely, when we have only voter individuals, i.e. $\rho_2 \to 1$ then $m \to 0$ and the boundary tends to be horizontal. In figure 12 we plot the angular coefficient of the boundary line as a function of $\rho_2 \in [0, 1]$.

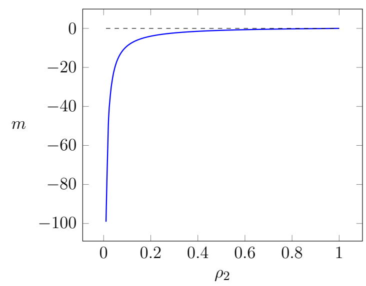
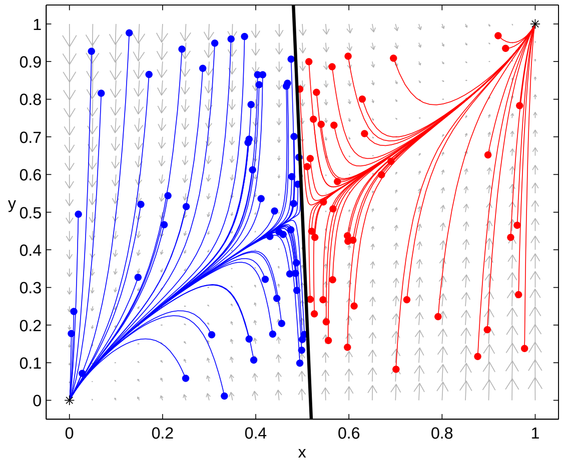
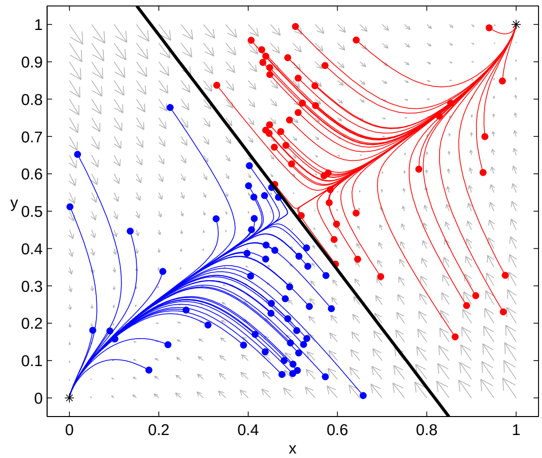


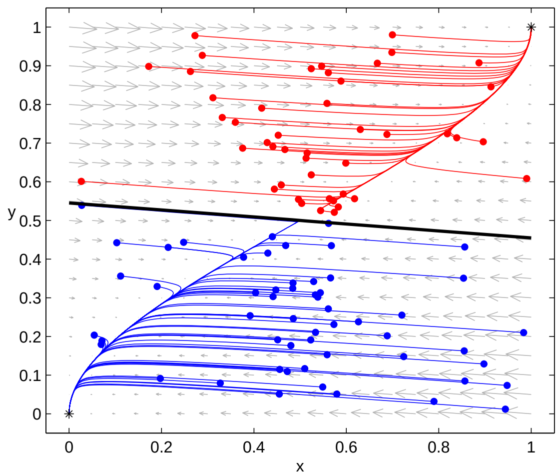
Figure 12: Angular coefficient of the boundary of the basins of attractions.



(a) Almost all majority individuals ($\rho_2 = 0.05$).



(b) Population split in two ($\rho_2 = \rho_1 = 0.5$).



(c) Almost all voter individuals ($\rho_2 = 0.95$).

Figure 13: The two extreme and intermediate cases of the model.

From the figure is possibile to notice that, when we have almost all majority individuals (case (a)), then as we could expect, the only thing that matters is the dominant opinion among them, regardless of the voters opinion. In other words, the dominant opinion in the majority population wins, so that if the majority agents have more ones than zeros, the population will achieve consensus in all-ones-state, otherwise the system will converge towards the all-zeros equilibrium.

The same, but with the voter individuals, happens in the other extreme case (c), when we have almost all voters. In this case of course, will be the voters' dominant opinion to prevail. In this case, we can see from the picture that for a while, as we could expect, we don't appreciate any dynamic of the voters opinion (y is almost costant) like in the classic voter model. After same time, the trajectory meets the unstable manifold of the central saddle equilibrium and then converges to a consensus.

The intermediate case (b) is when $\rho_2 = \rho_1 = \frac{1}{2}$. For these values, and if $q_1 = q_2$, we have $m = -1$ and the equation of the boundary line boils down to $y = 1 - x$.

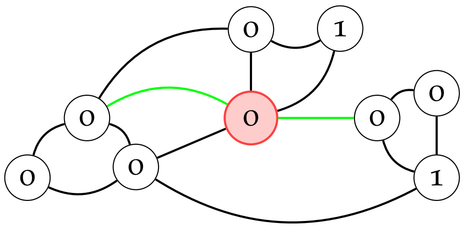
Mixed majority-voter model dynamics

As we have already pointed out, this model does not present anti social individuals and the analysis that we have carried out shows indeed a perfect match between the mean field model dynamics and the expected behaviour of the Markov process. The stability analysis has shown that the system admits the two pure configurations as asymptotically stable equilibria for every $\rho_2 \in (0, 1)$ and the system always converges to such points as no other stable attractors exist as proven in proposition 3.4.1. The boundary of the two basins of attraction is a line whose angular coefficient depends on q_1, q_2, ρ_2 in a complicate manner as shown in (95). Finally, we have seen that in the extreme cases, that is, when the population is composed by almost all voters or all majority individuals, the system goes towards the consensus point characterized by the preferred initial opinion of the dominant population.

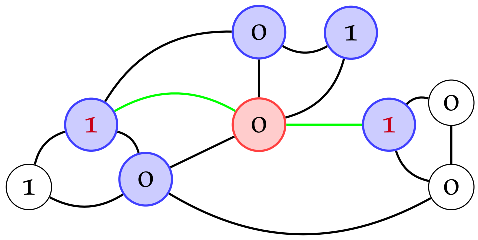
4

EDGE-BASED HETEROGENEOUS MODELS

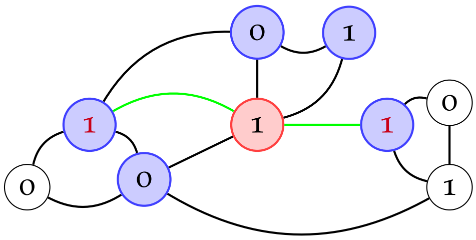
In this part we want to consider a differend kind of heterogeneous model. Similarly to the previous models, we will still consider two differend population and keep track of them with the same notations. However, if before we modelled the heterogeneity by considering a different dynamic for each population, in this case will be some edges between the two population that will generate a different kind of interaction between nodes if selected. In particular, we will consider as *heterogeneous edges* the ones connecting the two subpopulations. In other words, we are modeling a situation where the two subpopulations behave in a certain way as long as they interact within themselves through the "normal" edges, but when an individual of a population interacts with one of the others, then it behaves in a different way, in general following a different kind of dynamic. In some sense, this is a more general framework where each individual, and thus each population, has awareness of itslef, since it is able to recognize different people. Figure 14 shows an example involving a majority dynamic, where the green edges flips the opinion of the corresponding neighbours of the selected node.



(a) The red node is selected for an update.



(b) It looks at its neighbours, and it sees flipped the opinions of neighbours connected by the green edges.



(c) It selects the resulting local majority, in this case 1.

Figure 14: An example of majority dynamic with heterogeneous edges.

This kind of "edge-based" heterogeneity can hold for both population, and in that case we speak about symmetric heterogeneous edges, or instead it can affect only one population, in such case speak about asymmetrc heteorgeneous edges. We are going to investigate three edge-based heterogeneous models: the first one is characterized by symmetric heterogeneous edges, while the other two feature asymmetric heteorgeneous edges. Notice that for both cases, the interaction kernels of these kind of models can be described in the general form presented in chapter 2.

4.1 MAJORITY MODEL WITH SYMMETRIC ANTI MAJORITY EDGES

The first model with heterogeneous bonds that we consider involves the majority dynamic. Indeed, we consider two populations $\mathcal{V}_1$ and $\mathcal{V}_2$ following a 3-majority model. The heterogeneity is modelled as follows: when an agent of a certain population chooses an agent of the other population to interact with, it "sees" its opinion flipped. We can interpret this as a willing to differentiate the opinions between the two populations and thus, we can expect a polarization of opinions at equilibrium and not a global consensus.

In order to find the dynamic system describing the mean-field and hydrodynamic limit behaviour, we need to find the probabilities of increasing and decreasing the fractions of ones for each population. In the following, we evaluate in details the probability of increasing the fraction of ones in population $\mathcal{V}_1$, the others can be evaluated by means of the same argument.

To begin with, when an individual of $\mathcal{V}_1$ selects three other random people to interact with, four possible cases can occur: we can select none or one or two or three individuals belonging to $\mathcal{V}_2$ (we could also read: none or one or two or three anti-bonds). These cases are depicted in the following picture, where the colored nodes are those belonging to $\mathcal{V}_2$.

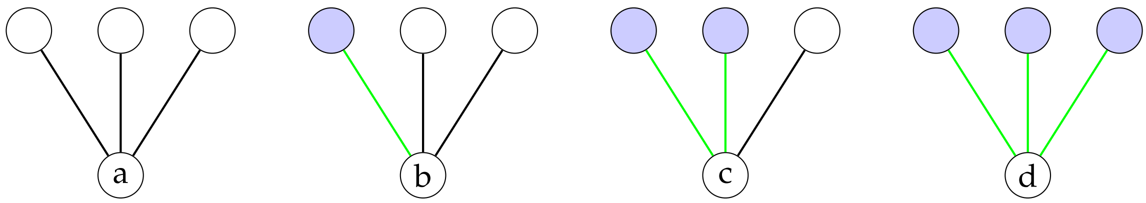


Figure 15: The four possible cases in the heterogeneous-bonds model.

Referring to the above picture, the probability to increase the number of ones of population $\mathcal{V}_1$ can be computed as: $\mathbb{P}(\text{ones increase}) = \mathbb{P}(\text{increase in (a)} \cup \text{increase in (b)} \cup \text{increase in (c)} \cup \text{increase in (d)})$. We have:

a. In this case we have all nodes belonging to population $\mathcal{V}_1$, which means that we have selected three times population $\mathcal{V}_1$. Then, in order do copy the one state, we need a majority of ones in the three nodes.

   We get: $\mathbb{P}(\text{increase in (a)}) = \rho_1^3(1-x)(3x^2(1-x)+x^3)$.

b. This time we have selected two nodes from $\mathcal{V}_1$ and a node belonging to $\mathcal{V}_2$, which means that we have an anti edge. We can compute the probability to copy state one by conditioning on the state of the V_2-node.

   We get: $\mathbb{P}(\text{increase in (b)}) = 3\rho_1^2\rho_2(1-x)((1-y)(1-(1-x)^2)+yx^2)$.

c. This time we have two anti link and a normal one and it is convenient to compute the probability by conditioning on the state of the$\mathcal{V}_1$-node.

   We get: $\mathbb{P}(\text{increase in (c)}) = 3\rho_1\rho_2^2(1-x)((1-x)(1-y)^2+x(1-y^2))$.

d. Finally, we have the case where we select only anti links, thus we need a majority of zeros (or a minority of ones, which is the same of course) to copy opinion one.

   We get: $\mathbb{P}(\text{increase in (d)}) = \rho_2^3(1-x)(3y(1-y)^2+(1-y)^3)$.

Finally, we can compute the probability to increase the number of ones in population $\mathcal{V}_1$ as follows:

$$\begin{aligned}\mathbb{P}(\text{ones incr. in } \mathcal{V}_1) &= \mathbb{P}(\text{incr. in (a)} \cup \text{incr. in (b)} \cup \text{incr. in (c)} \cup \text{incr. in (d)}) \\ &= \mathbb{P}(\text{incr. in (a)}) + \mathbb{P}(\text{incr. in (b)}) + \mathbb{P}(\text{incr. in (c)}) + \mathbb{P}(\text{incr. in (d)}) \\ &= (1-x)(\rho_2 + \rho_1 x - \rho_2 y)^2(\rho_2 + \rho_1(3-2x) + 2\rho_2 y)\end{aligned}$$

We can use the same argument to find the probability to decrease the fraction ones in $\mathcal{V}_1$ and the same can be done for population $\mathcal{V}_2$. Doing that and simplifying the equations, we obtain:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -x + (x - \rho_2(x+y-1))^2(3 - 2x + 2\rho_2(x+y-1)) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 - y - (x - \rho_2(x+y-1))^2(3 - 2x + 2\rho_2(x+y-1)) \end{cases} \tag{95}$$

Stability analysis

In order to find the equilibria we must solve

$$\begin{cases} -x + (x - \rho_2(x+y-1))^2(3 - 2x + 2\rho_2(x+y-1)) = 0 \\ 1 - y - (x - \rho_2(x+y-1))^2(3 - 2x + 2\rho_2(x+y-1)) = 0 \end{cases} \tag{96}$$

With some effort one can show that the system admits the following three solutions:

$$(x_{eq},\, y_{eq})_{\mathrm{I}} = \;(0,1) \qquad (x_{eq},\, y_{eq})_{\mathrm{II}} = \left(\frac{1}{2},\, \frac{1}{2}\right) \qquad (x_{eq},\, y_{eq})_{\mathrm{III}} = \;(1,0) \tag{97}$$

Let us now check the stability properties of these points.

Equilibrium I

The jacobian matrix is:

$$J_I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{98}$$

and we immediately see that the two eigenvalues are equal to -1, so equilibrium I is asintotically stable.

Equilibrium II

The jacobian matrix is:

$$J_{II} = \begin{bmatrix} \frac{1}{2}(1-3\rho_2) & -\frac{3}{2}\rho_2 \\ -\frac{3}{2}(1-\rho_2) & \frac{3}{2}\rho_2 - 1 \end{bmatrix} \tag{99}$$

and we see that:

$$\text{Tr}(J) - \frac{1}{2} < 0 \qquad \text{Det}(J) = -\frac{1}{2} < 0$$

This implies that the jacobian has a negative and a positive eigenvalue, which means that the point is an unstable saddle.

Equilibrium III
The results are the same as equilibrium I, thus the point is asintotically stable.

As we could expect, the stable equilibria correspond to the two polarization points where the two populations reach local consensus. Furthermore, those points are the only attractors for the system, as we prove in the following.

Proposition 4.1.1 *The three equilibria I, II and III are the only limit sets for the system.*

Proof *By Poincare'-Bendixson theorem, we just need to rule out the presence of limit cycles and in order to do that we use the Bendixson-Dulac theorem.*
Computing the divergence of the vector field we can see that :

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -2(1 + 3(x-1)x - 3\rho_2(2x-1)(x+y-1) + 3\rho_2^2(x+y-1)^2) \quad (100)$$

Notice that we can say anything just looking at the signs. However, we can notice that the function of two variables x and y defined by (100) on the compact set $[0,1]^2$ attains its global maximum in it by Weierstrass theorem. Moreover, we can see that the divergence is a concave function.
Hence, one can check that this global maximum is $-\frac{1}{2}$ by solving a convex optimization problem, that is:

$$\begin{cases} \max\limits_{x,y} & \left[\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}\right] \\ s.t & 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \end{cases} = -\frac{1}{2}$$

This implies that the divergence is always negative and non zero in its domain of definition. By Bendixson-Dulac theorem, no limit cycles can exist in the domain, which in turn implies that equilibria I, II and III are the only attractors for the system because of Poincare'-Bendixson theorem. ∎

To complete the global stability analysis we need to charachterize the two basins of attractions. We use the usual argument of finding the stable space of the saddle equilbrium, which is our boundary. By computing the eigenvector of jacobian (99), we see that the stable space is given by the following line: $L_I = \{v \in \mathbb{R}^2 : v = a(\frac{\rho_2}{1-\rho_2}, 1)^T,\ a \in \mathbb{R}\}$, from which we find the equation of the boundary line:

$$y = \frac{1-\rho_2}{\rho_2}x + \frac{2\rho_2 - 1}{2\rho_2} \quad \Longrightarrow \quad \rho_2 y - \rho_1 x = \rho_2 - \frac{1}{2} \tag{101}$$

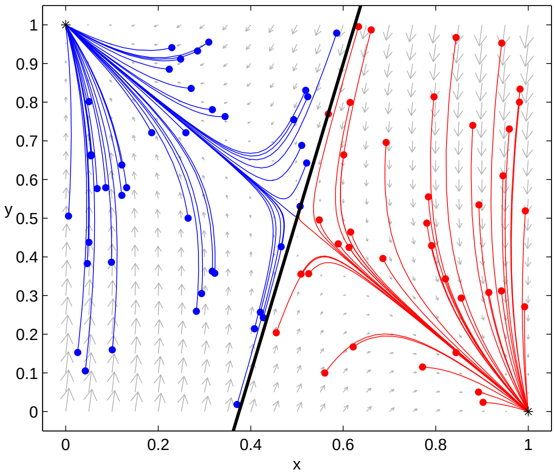
which can be re-written as:

$$\rho_2 y - \rho_1 x = \frac{1}{2}(\rho_2 - \rho_1) \tag{102}$$

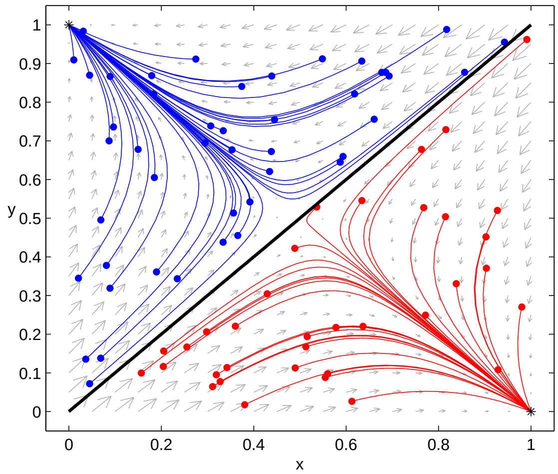
From the last equation we see that this locus of points has a simple interpretation: it is the locus of points where the difference between the two fractions of ones with respect to the whole population is proportional to the difference between the population fractions.

Given two initial fractions of ones x_0 and y_0 in the respective populations, we can thus say the following:

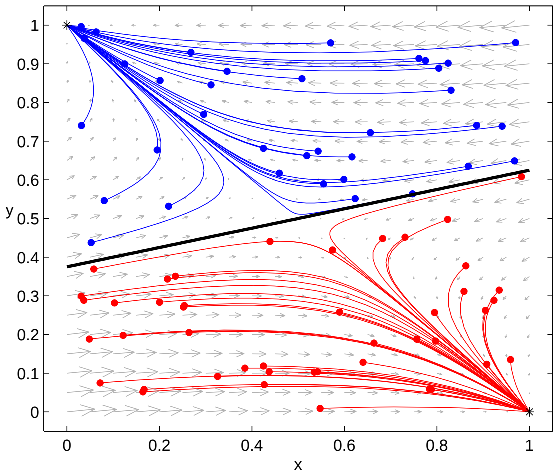
- If $\rho_1 x_0 - \rho_2 y_0 > \frac{1}{2}(\rho_2 - \rho_1)$, then the trajectory will converge to equilibrium III.
- If $\rho_1 x_0 - \rho_2 y_0 < \frac{1}{2}(\rho_2 - \rho_1)$, then the trajectory will converge to equilibrium II.



(a) $\rho_2 = 0.2$.



(b) $\rho_2 = \rho_1 = 0.5$.



(c) $\rho_2 = 0.8$.

Figure 16: Three portraits of the model for different values of ρ_2 simulated in MATLAB. We can appreciate the boundary of the basins of attractions.

Dynamics of the majority model with symmetric heterogeneous edges

The analysis that we have carried out shows that the model converges to one of the two polarization points of local consensus ($(0,1)$ and $(1,0)$) for any value of $\rho_2 \in (0,1)$; in particular, we do not have phase transitions. This kind of dynamic is coherent with the expected dynamic of the underlying Markov process.

Indeed, we can observe that the two polarization points are absorbing states for the Markov chain. The two basins of attractions of the polarization points are separated by the line (102), whose angular coefficient is give by $m = \frac{\rho_1}{\rho_2}$. In figure 16 we can observe three different portraits for different values of ρ_2. We can notice that, in the extreme cases $\rho_2 \to 0$ and $\rho_2 \to 1$, the boundary line tends to became vertical and horizontal respectively. This means that the dominant opinion of the largest subpopulation will prevail.

4.2 MAJORITY MODEL WITH ASYMMETRIC ANTI MAJORITY EDGES

In this section we consider a model with *asymmetric* heterogeneous edges, in the sense that only one population is affected by the anti-edges as it interacts with the other one. More in detail, we consider individuals belonging to population $\mathcal{V}_1$ following a plain 3-majority model, independently on who they interact with. Individuals belonging to population $\mathcal{V}_2$ instead, follows a 3-majority model locally and are affected by heterogeneous edges whenever they interact with population $\mathcal{V}_1$. This means that an agent in population $\mathcal{V}_2$ sees a $\mathcal{V}_1$-agent with its opinion flipped. This kind of model may be interpreted as the interaction between two populations, where one tries to reach a global consensus ($\mathcal{V}_1$) and the other one is more conservative and looks for a local consensus. Hence, we expect interesting threshold phenomena with respect to the parameter ρ_2.

Since the two dynamics involving the populations have been already studied in this dissertation, we already have the two equations that compose the model. Indeed, the mean field model is given by:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2\rho_1^3 x^3 - 3\rho_1^2(2\rho_2 y + 1)x^2 + (6\rho_1(1-\rho_2 y)\rho_2 y - 1)x + \rho_2^2 y^2(3 - 2\rho_2 y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 - y - (x - \rho_2(x+y-1))^2(3 - 2x + 2\rho_2(x+y-1)) \end{cases} \tag{103}$$

The next step is carrying out the stability analysis.

Stability analysis

In order to find the equilibria, we must solve

$$\begin{cases} -2\rho_1^3 x^3 - 3\rho_1^2(2\rho_2 y + 1)x^2 + (6\rho_1(1-\rho_2 y)\rho_2 y - 1)x + \rho_2^2 y^2(3 - 2\rho_2 y) = 0 \\ 1 - y - (x - \rho_2(x+y-1))^2(3 - 2x + 2\rho_2(x+y-1)) = 0 \end{cases} \tag{104}$$

Unfortunately, in this case is not possible to find the equilibria in closed form, with the only exception of the point $(x_{eq},\ y_{eq})_{\mathrm{I}} = (\frac{1}{2}, \frac{1}{2})$, which can be easly verified to solve the equilibrium equations for every value of ρ_2. With the help of the Mathematica software, one can numerically find the equilibria and we see that the non linear system of equations admits only equilibrium I as solution in a certain range of values of ρ_2; outside this range, the system admits three real solutions: one it is of course equilibrium I and other two real solutions different from equilibrium I, which will be called equilibrium II and III. Even without knowing the exact form of the solutions, by running simulations we can see equilibria II and III are always symmetric with respect to $y = 1 - x$. This can be stated analytically:

Proposition 4.2.1 *Let $(x^*, y^*) \in [0,1]^2$ be an equilibrium solution of system (104). Then, $(1 - x^*, 1 - y^*)$ is also a solution.*

Proof *The proof is straightforward. Indeed, by substituting $x \to 1 - x$ and $y \to 1 - y$ in system (), we obtain that:*

$$\begin{cases} -2\rho_1^3(1-x)^3 - 3\rho_1^2(2\rho_2(1-y)+1)(1-x)^2 \\ +(6\rho_1(1-\rho_2(1-y))\rho_2(1-y)-1)(1-x) + \rho_2^2(1-y)^2(3-2\rho_2(1-y)) = 0 \\ \\ 1-(1-y)-((1-x)-\rho_2(1-x-y))^2(3-2(1-x)+2\rho_2(1-x-y)) = 0 \end{cases} \tag{105}$$

After semplifying, the above system can be reduced to:

$$\begin{cases} -2\rho_1^3x^3 - 3\rho_1^2(2\rho_2 y+1)x^2 + (6\rho_1(1-\rho_2 y)\rho_2 y - 1)x + \rho_2^2y^2(3-2\rho_2 y) = 0 \\ \\ 1-y-(x-\rho_2(x+y-1))^2(3-2x+2\rho_2(x+y-1)) = 0 \end{cases} \tag{106}$$

which is the same as system (104). ∎

By running simulations, we can notice that equilibrium I is asymptotically stable in its range of existence and, outside it, it becomes unstable and the other two stable equilibria appear. In order to find informations about this bifurcation, we investigate the stability properties of this equilibrium.

The jacobian computed in equilibrium I is:

$$J_I = \begin{bmatrix} \frac{1}{2}(1-3\rho_2) & \frac{3}{2}\rho_2 \\ \frac{3}{2}(\rho_2 - 1) & \frac{3}{2}\rho_2 - 1 \end{bmatrix} \tag{107}$$

The eigenvalues are given by:

$$\lambda_1 = \frac{1}{4}\left(-1 - 3\sqrt{8\rho_2^2 - 8\rho_2 + 1}\right) \qquad \lambda_2 = \frac{1}{4}\left(-1 + 3\sqrt{8\rho_2^2 - 8\rho_2 + 1}\right) \quad (108)$$

We can study these two functions as ρ_2 varies in $[0, 1]$. The following picture shows both λ_1 and λ_2.

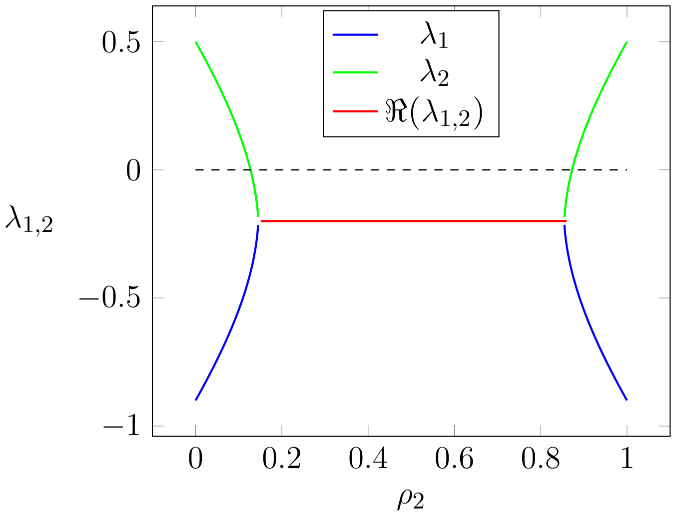


Figure 17: Eigenvalues λ_1 and λ_2 as functions of ρ_2.

The red part is the real part of the two eigenvalues when they are complex numbers for certain values of ρ_2. We can notice that λ_1 is always negative and so, the equilibrium point will be stable when also λ_2 is negative. We easily find that $\lambda_2 < 0 \iff \frac{1}{6}(3-\sqrt{5}) < \rho_2 < \frac{1}{6}(3+\sqrt{5})$. Thus, we can say the following:

$$(x_{eq}, y_{eq})_{\mathrm{I}} = \left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{is:} \quad \begin{cases} \text{unstable} & 0 < \rho_2 < \frac{1}{6}(3-\sqrt{5}) \text{ or } \frac{1}{6}(3+\sqrt{5}) < \rho_2 < 1 \\ \text{stable} & \frac{1}{6}(3-\sqrt{5}) < \rho_2 < \frac{1}{6}(3+\sqrt{5}) \end{cases} \quad (109)$$

We can see that the system admits two bifurcation points where the central equilibrium changes its stability properties. The first bifurcation occurs at $\rho_2 = \frac{1}{6}(3-\sqrt{5}) \approx 0.13$ and the second one at $\rho_2 = \frac{1}{6}(3+\sqrt{5}) \approx 0.87$. When equilibrium I is unstable, the other two stable equilibria appear. Notice that these thresholds are exactly the ones that define the range of existence of the equilibrium solutions. More in details, the situation can be resumed as follows:

- When $0 < \rho_2 < \frac{1}{6}(3-\sqrt{5})$ the system admits three equilibria: equilibrium I, which is unstable, and other two equilibria that are stable.

- When $\frac{1}{6}(3-\sqrt{5}) \leq \rho_2 \leq \frac{1}{6}(3+\sqrt{5})$ the system admits only equilibrium I, which is globally stable.
- When $\frac{1}{6}(3-\sqrt{5}) < \rho_2 < \frac{1}{6}(3+\sqrt{5})$ the system admits three equilibria: equilibrium I, which is unstable, and other two equilibria that are stable.

It is interesting to notice that there exist two range of values for ρ_2 in which the two eigenvalues associated to the jacobian (107) are both negative and reals. Indeed, by the eigenvalues expressions (46) we see that they are reals if and only if $8\rho_2^2 - 8\rho_2 + 1 \geq 0 \iff \rho_2 \leq \frac{1}{4}(2-\sqrt{2})$ or $\rho_2 \geq \frac{1}{4}(2+\sqrt{2})$.

Using the same thechinque we used for the previous models, it is possibile to approximate the boundary of the two basins of attractions, that appear in the case when equilibrium I is unstable, with the stable space of the equilibrium itself (which is a saddle point). In particular, we find that the angolar coefficient of the boundary line is:

$$m = \frac{2(1-\rho_2)}{2\rho_2 - 1 + \sqrt{8\rho_2^2 - 8\rho_2 + 1}} \tag{110}$$

The above formula holds for $0 < \rho_2 < \frac{1}{6}(3-\sqrt{5})$ and $\frac{1}{6}(3+\sqrt{5}) < \rho_2 < 1$. It can be noticed that when $\rho_2 \to 0 \Rightarrow m \to \infty$ and the boundary line tends to be vertical; conversely, when $\rho_2 \to 1 \Rightarrow m \to 0$ and the boundary line tends to be horizontal.

Dynamics of the majority model with asymmetric anti majority edges

The dynamics of the model can be resumed as follows in terms of values of ρ_2:

- $0 < \rho_2 < \dfrac{1}{6}(3-\sqrt{5})$

  In this case the system admits two stable equilibria, which are symmetric with respect to $y = 1 - x$. By running simulations, we can observe that such points are close to the two polarization points. This means that a quasi-local consensus is reached within each subpopulations.

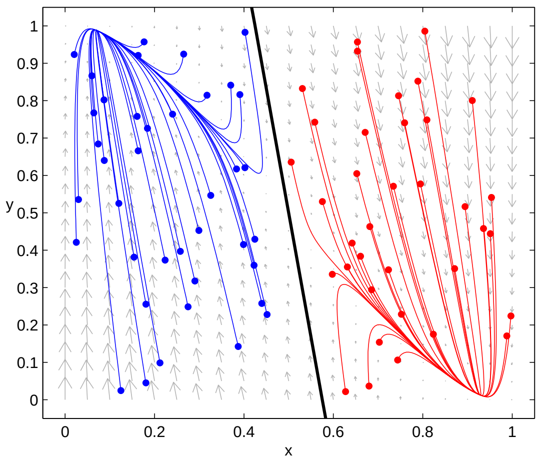
- $\dfrac{1}{6}(3-\sqrt{5}) \leq \rho_2 \leq \dfrac{1}{6}(3+\sqrt{5})$

  In this case the system admits only one equilibrium point (equilibrium I), which is globally stable. Notice that in this range the system converges towards the state of maximum entropy. As we have seen from the stability analysis, from a topological point of veiw, the convergence can occur in two ways: for $\frac{1}{6}(3-\sqrt{5}) < \rho_2 \leq \frac{1}{4}(2-\sqrt{2})$ the eigenvalues associated to equilibrium I are both negative and reals and the equilibrium is a stable node;

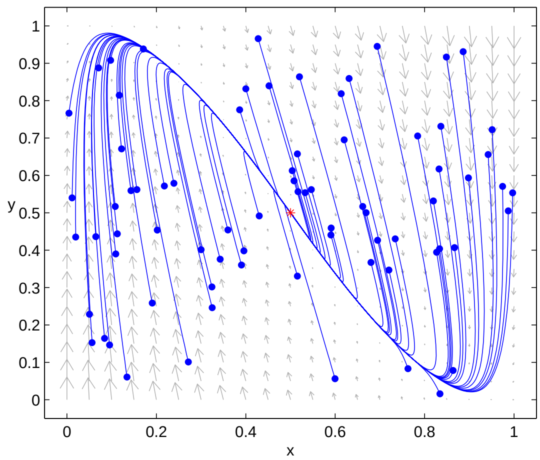
instead, for $\frac{1}{4}(2+\sqrt{2}) \leq \frac{1}{6}(3+\sqrt{5})$, the eigenvalues are both complex with negative real parts, thus the equilibrium is a stable focus.

- $\dfrac{1}{6}(3+\sqrt{5}) < \rho_2 < 1$
  In this case the system admits two stable equilibria, which are symmetric with respect to $y = 1 - x$. By running simulations, we can observe that such points are close to the two consensus points. This means that a quasi-global consensus is reached within the whole population.

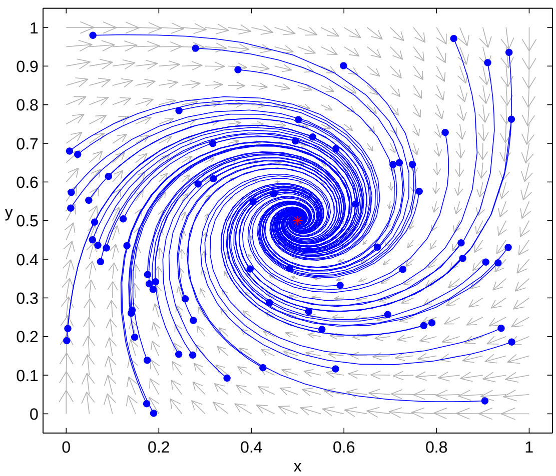
In figure 18 we show four portraits of the model for different values of ρ_2.



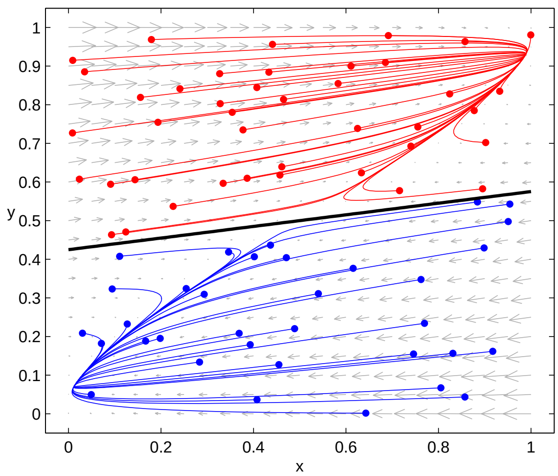
(a) $\rho_2 = 0.1$.



(b) $\rho_2 = 0.13$.



(c) $\rho_2 = 0.5$.



(d) $\rho_2 = \rho_1 = 0.9$.

Figure 18: Four portraits of the model for different values of ρ_2 simulated in MATLAB. We can appreciate the different kind of convergence towards equilibrium I (figure (b) and (c)) as well as the boundaries of the basins of attractions (figure (a) and (d)).

5

BEYOND THE MEAN FIELD AND KURT'S THEOREM

In this chapter we will consider two heterogeneous model that, for different reasons, goes beyond the "classical" models that we have seen so far. More in detail, in the first section we will consider a model involving a full majority dynamic, for which an individual copies the majority population observed among the whole population. We will see that the need to model this kind of dynamic will bring to a differential system with discontinuous right-hand side, a situation for which the Kurt's theorem does not hold.

In the second part of the chapter instead, we will drop the classical mean field assumption by considering an anti social dynamic taking place over a star graph. In particular, we will show some quasi-mean field results for such a model.

5.1 FULL MAJORITY MODEL WITH ASYMMETRIC HETEROGENEOUS EDGES

As already anticipated, in this section we are going to deal with a differential equation with discontinuous right-hand side. Thus, it is useful to introduce some basic theory needed to deal with such equations. In particular, we will consider the notion of solution in sense of Filippov. A complete analysis of this topic can be found in [10] and [12].

5.1.1 *Preliminaries on differential equations with discontinuous right-hand side*

There are several definitions of solutions for differential equations with discontinuous right-hand side and a complete discussion about this theory is far beyond the scope of this dissertation. For our purpose, however, it is enough to introduce one of the best known approach to this problem, which is based on the concept of *differential inclusion*.

We consider a generic system of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \tag{111}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise continuous function in a domain $\mathcal{D}$. Moreover, let $\mathcal{S}$ be a set (of measure zero) of points of discontinuity of the vector field f. The main idea of most of the known definition of solution is to describe how the vector field behaves around each point of the domain. In particular, this is necessary in order to deal with points belonging to the discontinuity set (around which the vector field will assume different values). This is done by defining a set-valued map $F(x)$, with $F : \mathbb{R}^n \to \mathcal{B}(\mathbb{R}^n)$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the collection of subset of $\mathbb{R}^n$. Just as a standard map or function takes a point in its domain to a single point in another space, a set-valued map takes a point in its domain to a set of points in another space. Furthermore, If at a point x the vector field f is continuous, then the set $F(x)$ consists of just one point which coincides with the value of the vector field f at this point. Coversely, in a point of discontinuity of f, the set $F(x)$ is given in some other way.

Then, a *solution* of the equation (58) is called a solution of the differential inclusion

$$\frac{\mathrm{d}x}{\mathrm{d}t} \in F(x) \tag{112}$$

that is, an absolutely continuous vector-valued function $x(t)$ defined on an interval or on a segment I for which $\frac{\mathrm{d}x(t)}{\mathrm{d}t} \in F(x(t))$ almost everywhere on I.

A differential inclusion thus specifies that the state derivative belongs to a set of directions, rather than being a specific direction. Of mayor interest are the methods of definition of the map F at the points of discontinuity of the function f, under which the above differential inclusion can be applied to approximate description of processes in real physical systems. In our case, we are going to construct the map F by means of the *Filippov set-valued map*, which is defined as follows:

Filippov set-valued map Given a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, we define the associated Filippov set-valued map $F[f] : \mathbb{R}^n \to \mathcal{B}(\mathbb{R}^n)$ as follows:

$$F[f](x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{\mathrm{co}}\{f(B(x,\delta))\backslash S\} \tag{113}$$

Here $\overline{\mathrm{co}}$ denotes the convex closure, μ denotes the Lebesgue measure and $B(x,\delta)$ is the ball centered in x with radius $\delta > 0$.

The idea under the definition (60), as already mentioned above, is to look at how the vector field looks like in a neighbourhood of a certain point (in particular, a point of discontinuity for f), and the map $F[f](x)$ is independent on the value of the vector field in x by construction. Specifically, for $x \in \mathbb{R}^n$, the vector field f is evaluated at the points belonging to $B(x,\delta)$. We examine the effect of δ approaching 0 by performing this evaluation for smaller and smaller δ. For additional flexibility, we exclude an arbitrary set of measure zero in $B(x,\delta)$ when evaluating f, so that the outcome is the same for two vector fields that differ on a set of measure zero (for instance, the value of the Heaviside function in the origin).

By means of the Filippov-set valued map definition, the equation (58) is replaced by the differential inclusion

$$\frac{\mathrm{d}x}{\mathrm{d}t} \in F[f](x) \tag{114}$$

An absolutely continuous vector-valued function $x(t)$ defined on an interval or on a segment I which satisfies the above differential inclusion almost everywhere on I, is called a *Filippov solution* of the equation (58). The next result establishes mild conditions under which Filippov solutions exist.

Theorem 5.1.1 (Existence of Filippov solutions) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be measurable and locally essentially bounded, that is, bounded on a bounded neighborhood of every point, excluding sets of measure zero. Then, for all $x_0 \in \mathbb{R}^n$, there exists a Filippov solution of (58) with initial condition $x(0) = x_0$.*

In most cases, like in the case of interest for us, the vector field f is piecewise continuous.

Piecewise-continuous vector fields The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous if there exists a finite collection of disjoint, open, and connected sets $D_1, \dots, D_m \subset \mathbb{R}^n$ whose closures cover $\mathbb{R}^n$, that is, $\mathbb{R}^n = \cup_{j \neg k=1}^{m} \overline{D_k}$, such that, for all $k = 1 \dots, m$, the vector field f is continuous on D_k. We further assume that the restriction of f to D_k admits a continuous extension to the closure $\overline{D_k}$, which we denote by $f_{\overline{D_k}}$. Every point of discontinuity of f must therefore belong to the union of the boundaries of the sets $D_1, \dots, D_m$. Let us denote by $\mathcal{S}_f \subseteq \partial D_1 \cup \dots \cup \partial D_m$ the set of points where f is discontinuous. Note that $\mathcal{S}_f$ has measure zero.

When f is piecewise continuous, the associated Filippov set-valued map can be written more explicitely as follows:

$$F[f](x) = \overline{\mathrm{co}} \left\{ \lim_{i \to \infty} f(x_i) \; : \; x_i \to x, x_i \notin \mathcal{S}_f \right\} \tag{115}$$

This is somehow more intuitive that the general definition (60). Notice that, at points of continuity for f, that is $x \notin \mathcal{S}_f$, we have $F[f](x) = f(x)$. At points of discontinuity of f, that is, for $x \in \mathcal{S}_f$, $F[f](x)$ is a convex polyhedron in $\mathbb{R}^n$ of the form

$$F[f](x) = \mathrm{co}\{ f_{\overline{D_k}}(x) \; : \; x \in \partial D_k \} \tag{116}$$

It is useful to state also a theorem of uniqueness of solutions in the sense of Filippov. The following result identifies sufficient conditions for uniqueness specifically tailored for piecewise continuous vector fields.

Theorem 5.1.2 (Uniqueness of Filippov solutions) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise continuous vector field, with $\mathbb{R}^n = D_1 \cup D_2$. Let $\mathcal{S}_f = \partial D_1 = \partial D_2$ be the set of points at which f is discontinuous, and assume that $\mathcal{S}_f$ is a C^2-manifold. Furthermore, assume that, for $i \in \{1, 2\}$, $f_{\overline{D_i}}$ is continuously differentiable on D_i and $f_{\overline{D_1}} - f_{\overline{D_2}}$ is continuously differentiable on $\mathcal{S}_f$. If, for each $x \in \mathcal{S}_f$, either $f_{\overline{D_1}}(x)$ points into D_2 or $f_{\overline{D_2}}(x)$ points into D_1, then there exists a unique Filippov solution of (58) starting from each initial condition.*

5.1.2 *Deduction and analysis of the model*

The model that we consider here is a variation of the model 4.2 that we have seen at the end of the last chapter. We consider again two populations: individuals belonging to $\mathcal{V}_1$ follow a full majority dynamic while the ones belonging to $\mathcal{V}_2$ follow a 3-majority dynamic. Moreover, we consider the presence of asymmetric heterogeneous edges connecting the two populations; these edges affect only individuals belonging to $\mathcal{V}_2$, who will see individuals belonging to $\mathcal{V}_1$ with flipped opinions, as already explained in the previous chapter.

The interpretation of such a model can be the following: on one side we have population $\mathcal{V}_1$, which is very democratic as it looks at the opinion of each individual and tries to reach a global consensus. On the other side, we have individuals belonging to population $\mathcal{V}_2$, who tries to achieve a local consensus by means of a 3-majority dynamic and at the same time they try to differentiate their opinions from those of $\mathcal{V}_1$-individuals.

In order to find the dynamic equation for population $\mathcal{V}_1$, we observe that in the full majority it is enough to know the fraction of ones within the whole population (or of zeros of course) to know the majority at each instant of time. In other words, an individual does not have to choose a group of random people first; it will simply assume opinion 1 if the fraction of ones is greater than $\frac{1}{2}$ and instead will take opinion 0 in the other case. Furthermore, we assume that in case of a tie (which means that the fraction of ones is exactly $\frac{1}{2}$), the individual will choose the opinion at random among the two. This means that the probability to choose opinion one (or zero), given the actual fractions of ones within the whole population, is a Heaviside step function of the fraction of ones within the whole population, which can be either 1 (ones majority), 0 (zeros majority) or $\frac{1}{2}$ (tie). We indicate such a function with

$$H\left((1-\rho_2)x+\rho_2 y-\frac{1}{2}\right) \tag{117}$$

Thus, dynamic equation for population $\mathcal{V}_1$ is:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=(1-x)H\left((1-\rho_2)x+\rho_2 y-\frac{1}{2}\right)-x\left(1-H\left((1-\rho_2)x+\rho_2 y-\frac{1}{2}\right)\right)\\&=H\left((1-\rho_2)x+\rho_2 y-\frac{1}{2}\right)-x\end{aligned} \tag{118}$$

Then, remembering the equation for population V_2 from the previous model, the equations describing the mean-field model are given by:

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=H\left((1-\rho_2)x+\rho_2 y-\dfrac{1}{2}\right)-x\\[2mm]\dfrac{\mathrm{d}y}{\mathrm{d}t}=1-y-(x-\rho_2(x+y-1))^2(3-2x+2\rho_2(x+y-1))\end{cases} \tag{119}$$

Since the Heaviside function $H(\cdot)$ is discontinuos in the origin, the analysis of the above system needs some extra care. To begin with, we need to understand how solutions of the above system have to be interpreted. In particular, we will use the definition of *Filippov solution* that we have introduced before.

To begin with, we can notice that the vector field is piecewise continuous. In particular, the domain of definition of the vector field f can be decomposed into the two sub-domains in which f is continuous: $f:[0,1]^2\to\mathbb{R}^2$ with $[0,1]^2=D_1\cup D_2$ and

$$\mathcal{S}_f=\partial D_1=\partial D_2=\left\{(x,y)\in[0,1]^2\ :\ \rho_1 x+\rho_2 y=\frac{1}{2}\right\} \tag{120}$$

is the discontinuity set. Moreover, let us call f_y the y-component of the vector field (notice that this component is continuous in the whole domain and its continuous extension coincides with the value of component itself evaluated on the discontinuity set); then, the two continuous extensions of f to the closure of D_1 and D_2 are respectively

$$f_{\overline{D_1}}=f_1=\begin{bmatrix}-x\\f_y(\mathcal{S})\end{bmatrix}\qquad f_{\overline{D_2}}=f_2=\begin{bmatrix}1-x\\f_y(\mathcal{S})\end{bmatrix} \tag{121}$$

Moreover, in the following, let f_1^N and f_2^N be the projections of the vectors f_1 and f_2 onto the normal to the line defined by $\mathcal{S}$; the normal is directed towards the domain D_2.

The solution of system (52) is understood as solution of the following differential inclusion

$$\frac{\mathrm{d}X}{\mathrm{d}t} \in \begin{cases} f_{D_1} & (x,y) \in D_1 \\ \{\alpha f_1 + (1-\alpha) f_2 \;:\; \alpha \in [0,1]\} & (x,y) \in \mathcal{S} \\ f_{D_2} & (x,y) \in D_2 \end{cases} \tag{122}$$

Where for sake of notation we have grouped variables x and y into the vectorial variable X.

Proposition 5.1.3 *There exists a Filippov solution of the differential system (52)* $\forall\, \rho_2 \in (0,1)$ *and with initial condition* $x(0) = x_0,\ y(0) = y_0$*, for every* $(x_0, y_0) \in [0,1]^2$.

Proof *The vector field of system (52) is bounded and measurable* $\forall\, \rho_2 \in (0,1)$*. Then the existence of a Filippov solution for every initial condition is proven by means of theorem 3.4.* ∎

For what concerns uniqueness, some extra care is needed. In particular, we can observe that, for any value of ρ_2, there exist points (x,y) belonging to the discontinuity set $\mathcal{S}$ for which $f_{\overline{D_1}}(x,y)$ and $f_{\overline{D_2}}(x,y)$ point into D_1 and D_2 respectively (figure 13). This means that $f_1^N < 0$ and $f_2^N > 0$ and theorem 3.5 cannot be applied. At those points, the discontinuity set $\mathcal{S}$ is "repulsive", in the sense that the vector field brings a trajectory starting near $\mathcal{S}$ away from the set itself. Thus, the system admits two different Filippov solutions starting from such a points, known as *replulsive solutions*. Infact, in this case a trajectory may leave $\mathcal{S}$ under the effect of f_1 or f_2.

For instance, let us consider the case $\rho_2 < \frac{1}{2}$, for which we have repulsive solutions for every point belonging to the discontinuity set. Let us call for brevity f_y the y-component of the vector field of system (52) (notice that this component is continuous in the whole domain). Then, the solutions of the two following systems are both a Filippov solution for system (52) with initial condition $(x_0, y_0) \in \mathcal{S}$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = f_y \end{cases} \qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 1 - x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = f_y \end{cases} \tag{123}$$

After the bifurcation point, that is, for $\rho_2 > \frac{1}{2}$, we still have presence of repulsive solutions (as it is clear by looking at the vector fields (b) and (c) of figure (13)), for which we lose uniqueness. In this case, we also have presence of the so called *transversal intersection*, for which a trajectory passes from one side of the line $\mathcal{S}$

to the other. This happens for those points $(x, y) \in \mathcal{S}$ such that $f_1^N \cdot f_2^N > 0$, that is, the two vector field restrictions point to the same direction. Notice that the uniqueness of the Filippov solution starting from these points is guaranteed by means of theorem 3.5.

In particular, the following results hold:

Proposition 5.1.4 *System (119) admits only repulsive solutions starting from the discontinuity set $\mathcal{S}$ for $0 < \rho_2 < \frac{1}{2}$.*

Proof *To prove this statement we need to show that:*

$$f_1^N < 0 \text{ and } f_2^N > 0 \qquad \forall\, (x, y) \in \mathcal{S},\ \rho_2 < \frac{1}{2} \tag{124}$$

If we call $\phi = \rho_1 x + \rho_2 y - \frac{1}{2}$, we notice that our discontinuity line is given by the equation $\phi = 0$. The two projections are given by the equations

$$f_1^N = \frac{\nabla\phi \cdot f_1}{|\nabla\phi|} \qquad\qquad f_2^N = \frac{\nabla\phi \cdot f_2}{|\nabla\phi|} \tag{125}$$

From the above formula, after some algebra one finds:

$$f_1^N = \frac{-1 - \rho_2(-\rho_2 + 2(-1 + \rho_2)x)(3 - 2\rho_2 + 4(\rho_2 - 1)x)^2}{\sqrt{4 + 8(\rho_2 - 1)\rho_2}} \tag{126}$$

$$f_2^N = \frac{1 - \rho_2(2 - \rho_2 + 2(1 - \rho_2)x)(1 - 2\rho_2 + 4(1 - \rho_2)x)^2}{\sqrt{4 + 8(1 - \rho_2)\rho_2}} \tag{127}$$

A study of the above functions shows that $f_1^N < 0$ and $f_2^N > 0 \quad \forall\, (x, y) \in \mathcal{S},\ 0 < \rho_2 < \frac{1}{2}$ and the result is proven. ∎

Proposition 5.1.5 *System (119) admits transversal intersections for $\rho_2 > \frac{1}{2}$.*

Proof *To prove this statement it is enough to exhibit a transversal intersection for every $\rho_2 > \frac{1}{2}$. In particular, we need to find a point belonging to the discontinuity set for which the following holds:*

$$f_1^N f_2^N > 0 \qquad \text{for } \frac{1}{2} < \rho_2 < 1 \tag{128}$$

For instance, we can evaluate projections (126) and (127) at $x = 1$ (the y coordinate will be such that the point belongs to $\mathcal{S}$.), we obtain:

$$f_1^N f_2^N = \frac{(1 - \rho_2)^2(-1 + \rho_2(1 + 4(\rho_2 - 2)\rho_2)(1 + \rho_2(1 + 4(\rho_2 - 2)\rho_2))}{(4 + 8(\rho_2 - 1)\rho_2)} \tag{129}$$

A quick study of the above function shows that $f_1^N f_2^N > 0 \qquad \frac{1}{2} < \rho_2 < 1$ and the result is proven. ∎

To conclude, we can say that for every $\rho_2 \in (0,1)$ the differential system (119) admits a unique Filippov solution $\forall\,(x_0, y_0) \notin \mathcal{S}$ while the solution is in general not unique for $(x_0, y_0) \in \mathcal{S}$.

Stability analysis

We can start noticing that, at equilibrium, the first equation imposes: $H((1-\rho_2)x + \rho_2 y - \frac{1}{2}) = x$. Since H can only be either 0, $\frac{1}{2}$ or 1, this means that x can be either 0, $\frac{1}{2}$, or 11 as well. If we consider the case $x = 0$, the second equation becomes:

$$1 - \rho_2^2(3 + 2\rho_2(y-1))(y-1)^2 - y = 0 \iff \begin{cases} y = 1 \\ y = 1 - \dfrac{3\rho_2^2 + \sqrt{\rho_2^3(9\rho_2 - 8)}}{4\rho_2^3} \\ y = 1 - \dfrac{3\rho_2^2 - \sqrt{\rho_2^3(9\rho_2 - 8)}}{4\rho_2^3} \end{cases} \tag{130}$$

We notice that the last two solutions exist if and only if $\rho_2 \geq \frac{8}{9}$. Now we get back to the first equation, that evaluated for $x = 0$ becomes $H(\rho_2 y - \frac{1}{2}) = 0 \iff \rho_2 y - \frac{1}{2} < 0$. We have to check if this inequality is satisfied for the values of y found in (130). By substituting, we see that $y = 1$ is actually a solution if and only if $\rho_2 < \frac{1}{2}$. By doing the same for the other two values of y, we see that the inequality is always satisfied for $\rho_2 \geq \frac{8}{9}$, which is the range of existence of those values.

The other case we have to consider is when $x = 1$. For this value, the second equation is:

$$y(\rho_2^2 y(3 - 2\rho_2 y) - 1) = 0 \iff \begin{cases} y = 0 \\ y = \dfrac{3\rho_2^2 - \sqrt{\rho_2^3(9\rho_2 - 8)}}{4\rho_2^3} \\ y = \dfrac{3\rho_2^2 + \sqrt{\rho_2^3(9\rho_2 - 8)}}{4\rho_2^3} \end{cases} \tag{131}$$

As we could expect by symmetry reasons, we see that these solutions are just the one-complementary of those found for $x = 0$ in (130). The first equation evaluated for $x = 1$ gives $H(1 - \rho_2 + \rho_2 y - \frac{1}{2}) = 1 \iff \rho_2 y - \rho_2 + \frac{1}{2} > 0$. The conclusions are the same as before: $y = 0$ is a solution for $\rho_2 < \frac{1}{2}$ and the other two are solutions in their range of existence, $\rho_2 \geq \frac{8}{9}$.

Finally, we consider the case $x = \frac{1}{2}$. This means that the argument of the Heaviside function is 0, so it must be: $\frac{(1-\rho_2)}{2} + \rho_2 y - \frac{1}{2} = 0 \Rightarrow y = \frac{1}{2}$. By plugging $x = \frac{1}{2}$ and $y = \frac{1}{2}$ into the second equation, we see that this is actually a solution $\forall \rho_2 \in (0, 1)$.

Summing up our results, the system admits the following equilibria:

$$(x_{eq},\, y_{eq})_{\mathrm{I}} = (0, 1) \quad (x_{eq},\, y_{eq})_{\mathrm{II}} = (1, 0) \quad (x_{eq},\, y_{eq})_{\mathrm{C}} = \left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{for} \quad \rho_2 < \frac{1}{2}$$

and

$$(x_{eq}\, y_{eq})_{\mathrm{III}} = \left(0,\ 1 - \frac{3\rho_2^2 + \sqrt{\rho_2^3(9\rho_2 - 8)}}{4\rho_2^3}\right)$$
$$(x_{eq}\, y_{eq})_{\mathrm{IV}} = \left(0,\ 1 - \frac{3\rho_2^2 - \sqrt{\rho_2^3(9\rho_2 - 8)}}{4\rho_2^3}\right)$$
$$(x_{eq}\, y_{eq})_{\mathrm{V}} = \left(1,\ \frac{3\rho_2^2 + \sqrt{\rho_2^3(9\rho_2 - 8)}}{4\rho_2^3}\right)$$
$$(x_{eq}\, y_{eq})_{\mathrm{VI}} = \left(1,\ \frac{3\rho_2^2 - \sqrt{\rho_2^3(9\rho_2 - 8)}}{4\rho_2^3}\right)$$
$$(x_{eq},\, y_{eq})_{\mathrm{C}} = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \text{for} \quad \rho_2 \geq \frac{8}{9}$$

To complete the stability analysis we can check that all the equilibria that we have found are asyntotically stable in their range of existence, apart from equilibria III and IV and the central equilibrium that we have called C. In particular, keeping in mind that the partial derivatives of H are 0 almost everywhere, it is immediate to see that, for equilibria I and II, the jacobian is given by:

$$J_I = J_{II} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{132}$$

and we immediately see that the two eigenvalues are equal to -1, so equilibria I and II are asintotically stable.

Lenghtly but easy calculations show that equilibria V and VI are stable as well in their range of existence ($\rho_2 \geq \frac{8}{9}$), while equilibria III and IV are unstable.

We have some issues when we have to estabilish the stability condition of equilibrium C. In that case infact, we cannot simply consider the jacobian of the sistem evaluated in that point because the Heaviside function is not continuous in the origin and so we cannot take partial derivatives, at least in a classical sense.

However, one can check that the discontinuity line is always repulsive around equilibrium C. This implies that we can always exhibit solutions starting arbitrarly close to the equilibrium that do not satisfy the stability condition. Thus, equilibrium C is unstable for any value of $\rho_2 \in (0, 1)$.

From this analysis we can notice that there exists a range $\frac{1}{2} < \rho_2 < \frac{8}{9}$ in which the system does not admit any stable equilibrium. Since we know that our domain $[0, 1]^2$ is an invariant set, if the differential system were continuous, by Poincae-Bendixson theorem we could conlcude that a stable limit-cycle must have existed in that range. Unfortunately, the presence of a discontinuity makes this argument invalid as the Poincare-Bendixon theorem does not hold. However, by running simulations we will see that a periodic orbit appears for $\rho_2 > \frac{1}{2}$.

It is useful at this point to show how the vector field behaves for different values of ρ_2 (figure 19 to 21).

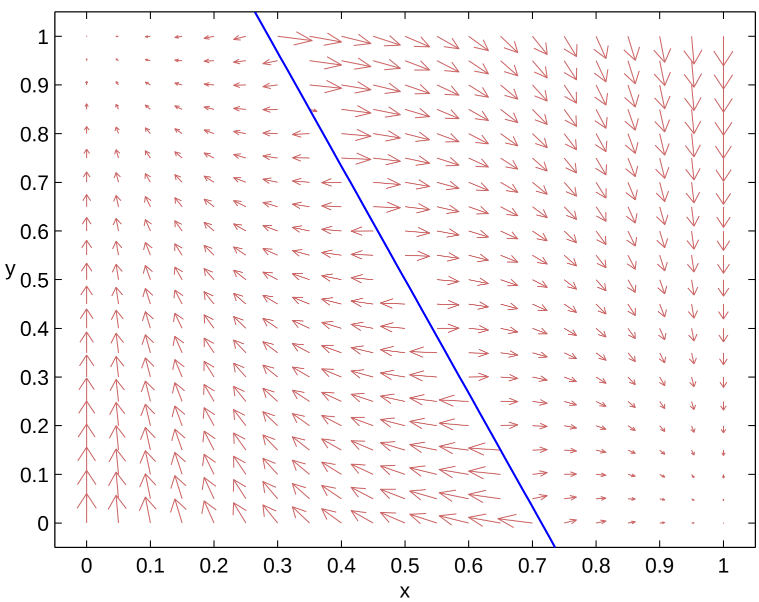


Figure 19: Vector field of the model for $\rho_2 = 0.3$. The blu line is the discontinuity set. We can appreciate that for this value fo ρ_2 the discontinuity set $\mathcal{S}$ is repulsive.

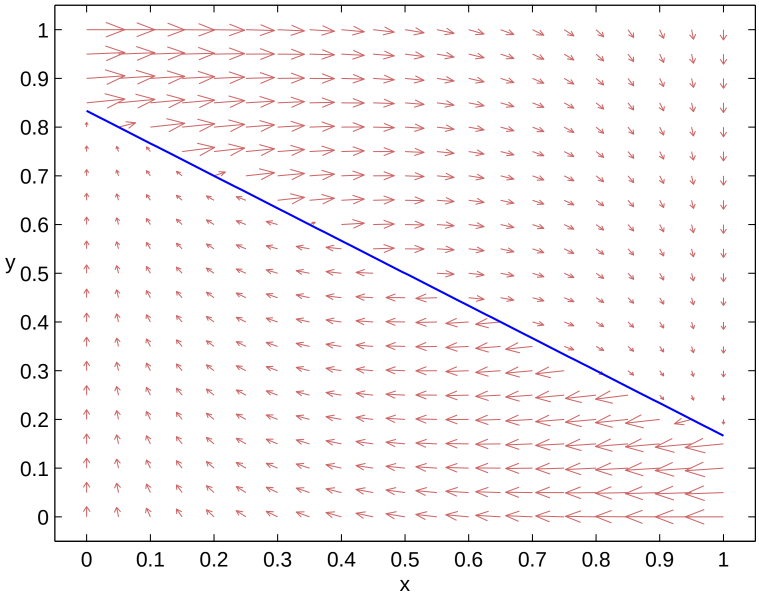


Figure 20: Vector field of the model for $\rho_2 = 0.6$. The blu line is the discontinuity set.

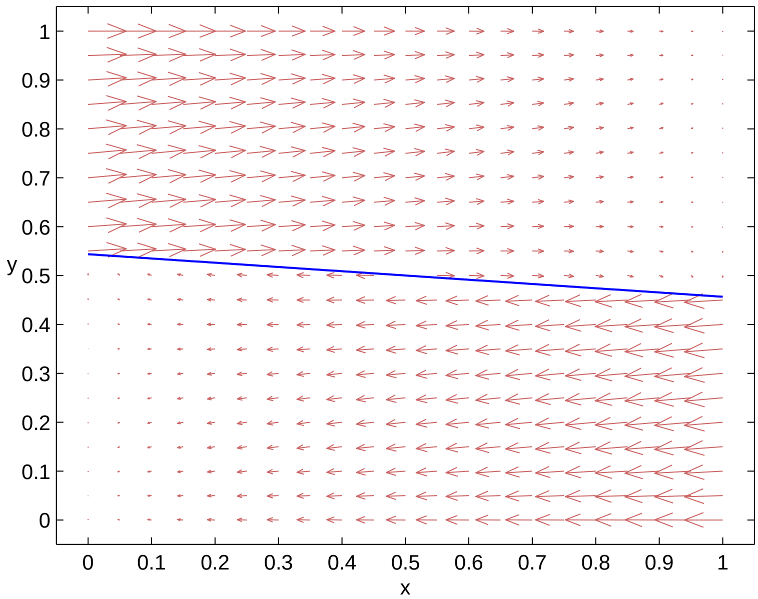


Figure 21: Vector field of the model for $\rho_2 = 0.92$. The blu line is the discontinuity set.

In the following, we will simulate in MATLAB the dynamics of the system for different values fo ρ_2.

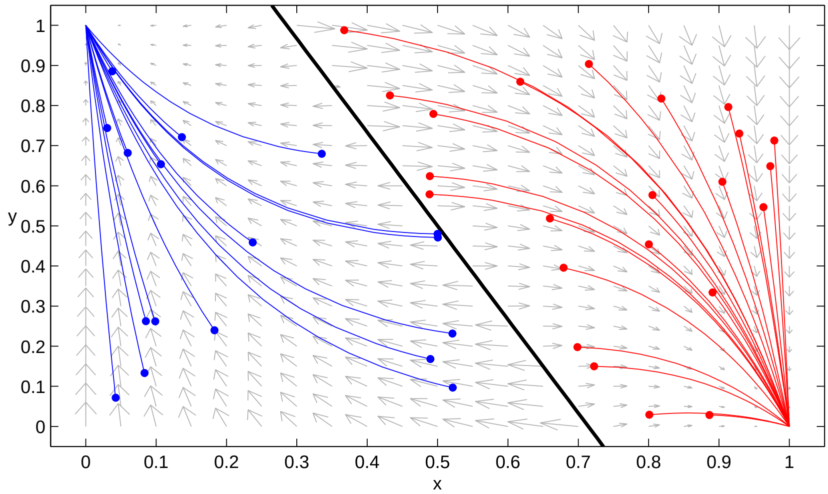


Figure 22: Some trajectories of the model for $\rho_2 = 0.3$ simulated in MATLAB. In black the discontinuity set.

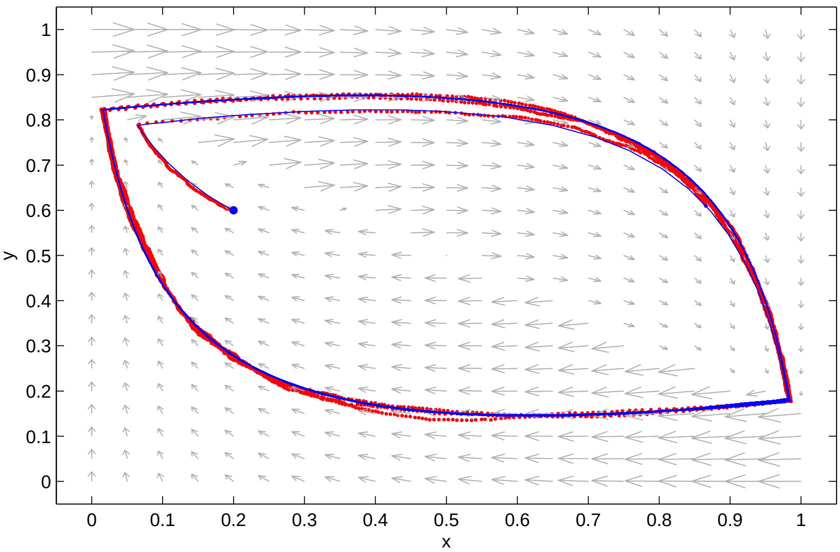


Figure 23: A trajectory of the model for $\rho_2 = 0.6$ simulated in MATLAB. We can notice that a limit cycle appears. In red, we can see a sample path of the underlying Markov process.

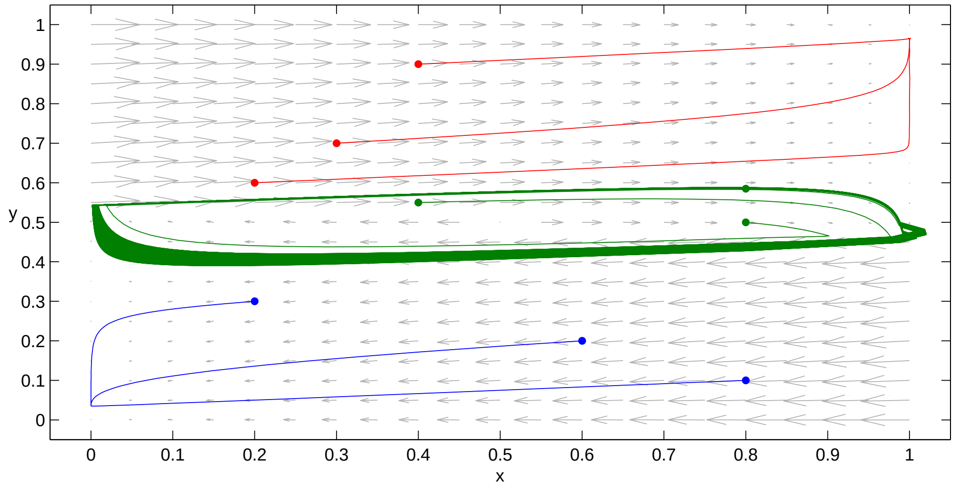


Figure 24: Some trajectories of the model for $\rho_2 = 0.92$ simulated in MATLAB. Two stable equilibria appear and the limit cycle is still present.

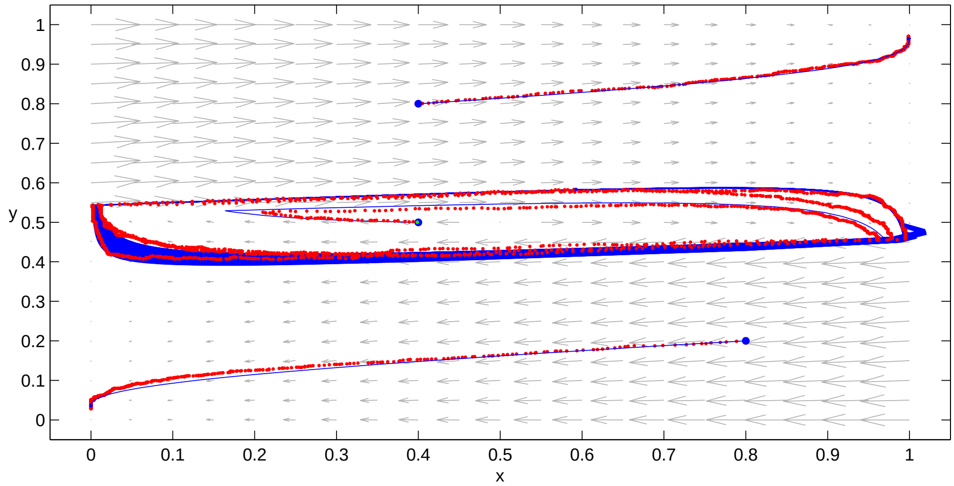


Figure 25: Some trajectories of the model for $\rho_2 = 0.92$ simulated in MATLAB. In red, we can see the corresponding sample paths of the Markov chain.

It is interesting to notice that the stable limit cycle exists even for $\rho_2 \geq \frac{8}{9}$, when also the two stable equilibria appear (figure 24). From simulations we can also see that the limit cycle shrinks as ρ_2 increases, up to collapse in the limit case $\rho_2 = 1$. In particular, figure 24 shows that, for $\rho_2 \geq \frac{8}{9}$, the cycle appears like a ellipsoid-shaped object with its "semi-major axis" perpendicular to axis y. This means that, when we have a large majority of anti-social individuals, if they start from a very uncertain and chaotic condition (which means without a predominant opinion among them), then they will bring the system to a periodic oscillation describing the limit cycle, regardless of the social-individuals opinion; otherwise, the population will reach one of the two equilibria, very close to consensus.

Dynamics of the full majority model with asymmetric heterogeneous edges

We can resume the dynamics of the model as follows:

- $0 < \rho_2 \leq \dfrac{1}{2}$
  In this case the system admits the two polarization points $(0, 1)$ and $(1, 0)$ as stable equilibria. This means that a quasi-local consensus is reached within each subpopulations.
- $\dfrac{1}{2} < \rho_2 \leq \dfrac{8}{9}$
  In this case the system admits no stable equilibria and a limit cycle appears. In this range, the opinions tend to oscillate in a periodic fashion.
- $\dfrac{8}{9} < \rho_2 < 1$
  In this case the stable equilibria V and VI appear and the stable limit cycle is still present. Notice that equilibria V and VI are points close to the pure configurations.

5.2 ANTI SOCIAL BEHAVIOUR DYNAMICS ON A STAR GRAPH

In our study of hetherogeneous model, so far, we have only considered the mean-field assumption, deriving our results working on complete graphs. In general, it can be very hard to find exact results about the asymptotic behaviour of an heterogeneous model taking place on a non-complete graph. However, in some particular cases, it is possible to give some simple structure to the underlying network without loosing the capability to carry out an analytic analysis of the model.

Here we consider one of these "lucky cases", where the network's structure is given by a star graph, like the one shown in the following picture.

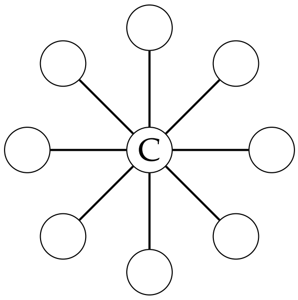


Figure 26: The star graph S_8.

A star graph S_n is nothing but a particular tree and bipartite graph, which consists of one central node, denoted by C, and a number n of periferic nodes connected to it. This structure, although very simple, can be used for instance to model social hierarchies, with the chief as the central node C and its subordinates as the periferic nodes.

Let us now describe the dynamic of the model. We consider a generic star-graph S_n and we split the n periferic nodes into two population as usual: $\mathcal{V}_1$ following a voter dynamic and $\mathcal{V}_2$ following an anti-voter dynamic. Finally, the central node C follows a 3-majority model.

We can derive a sort of hydrodynamic-limit model by considering a star-graph with a large number of periferic nodes (in the limit, $n \to \infty$). Since we have a very large amount of periferic nodes but still one central node, we need to assign a certain probability to select whether C or one of the periferic nodes for an opinion update at every discrete time step. In the following, we will denote by q_c the probability to pick up the central node for an update at each discrete time step (obvioulsy, $1 - q_c$ will be the probability to choose one of the periferic nodes). Moreover, we call $z(t)$ the probability that the central node is in state 1 at time t.

To begin with, the state of the node C at fixed instant of times t will be a random variable with bernoulli distribution, as it can assume only value 0 or 1. In particular, if we call such a variable $C(t)$, we have that:

$$C(t) = \begin{cases} 1 & \text{with probability} \quad z(t) \\ 0 & \text{with probability} \quad 1 - z(t) \end{cases} \tag{133}$$

where $z(t)$ is the probability that C is in state 1 at time t, as already mentioned before. Notice that the evolution in time of the state of C is described by a Markov process defined by the family of bernoulli variables $\{C_t,\ t \in \mathbb{R}^+\}$ with an initial conditions C_0. Notice also that the bernoulli variables are not independent nor identically distributed.

Finally, we can write a sort of hydrodynamic limit-model as two differential equations coupled by the stochastic process C_t:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = q(C_t - x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = q(1 - C_t - y) \end{cases} \tag{134}$$

It is clear that in this case the model is not deterministic anymore. However, we will see that the dynamics of the model are strongly affected by the value of q_c: More in details, when the probability to pick the node C q_c and the complementary probability to pick a periferic node are of the same order, then system (134) tends to behave in a deterministic fashion.

The case: $q_c \sim 1 - q_c$

In this scenario we can approximate the value of the random variable $C(t)$ at each fixed instant of time with its expected value: $\mathbb{E}[C(t)] = z(t)$. The equation for populations $\mathcal{V}_1$ can be written as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1 - q_c)((1 - x)z(t)q - x(1 - z(t))q) \tag{135}$$

where q is the copying probability. Doing the same for population $\mathcal{V}_2$ and simplifying, we obtain that the model can be written as follows:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = q(1 - q_c)(z(t) - x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = q(1 - q_c)(1 - z(t) - y) \end{cases} \tag{136}$$

Now we need an equation describing the evolution of the probability $z(t)$ involving the central node. It is rather simple to derive an equation for $z(t)$ thinking in terms of discrete time steps; we recall that the probability to find a majority of ones among three nodes chosen at random from N nodes is given by:

$$m = \sum_{i=2}^{3} \frac{\binom{N(\rho_1 x + \rho_2 y)}{i}\binom{1 - N(\rho_1 x + \rho_2 y)}{3-i}}{\binom{N}{3}} \tag{137}$$

Thus, by total probability theroem, the discrete-time equation for $z(k)$, (with $k \in \mathbb{N}$) can be written as follows:

$$z(k+1) = (1 - q_c)z(k) + q_c m(k) \tag{138}$$

Here the first term on the right-hand side represents the probability of the event: "C is in state 1 at time k and it is not selected for an update at time $k+1$"; the second term is simply the probability of the event: "C is selected at time $k+1$ and finds a ones-majority selecting three random nodes". The above equation can be rewritten as:

$$z(k+1) - z(k) = q_c(m(k) - z(k)) \tag{139}$$

This is our master equation for $z(k)$, which unfortunately has been derived in a different time scale with respect to the differential equations (136). We can use the same hydrodynamic-limit argument by changing the time scale accordingly. By doing that, and multipling both sides by N, we obtain:

$$\frac{z(t+\frac{1}{N}) - z(t)}{\frac{1}{N}} = Nq_c(m(t) - z(t)) \tag{140}$$

Now if we let $N \to +\infty$, the incremental ratio on the left-hand side will converge to the derivative of $z(t)$. We notice that there is an explicit dependence on N in the right-hand side of the equation. Then, we may write the following differential equation:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Nq_c(m(t) - z(t)) \tag{141}$$

which approximates the discrete equation (138) in continuous time when N is large. Coupling this equation with the system (136), the model can be approximated, for large N, by the following three-dimensional dynamical system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = q(1-q_c)(z-x) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = q(1-q_c)(1-z-y) \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = Nq_c(3((1-\rho_2)x + \rho_2 y)^2(1-(1-\rho_2)x - \rho_2 y) + ((1-\rho_2)x + \rho_2 y)^3 - z) \end{cases} \tag{142}$$

The equilibria of the above system can be computed analitically and they *do not* depend on the particular choise of q_c; lenghtly calculations show the they are given by:

$$(x_{eq},\, y_{eq},\, z_{eq})_{\mathrm{I}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$(x_{eq},\, y_{eq},\, z_{eq})_{\mathrm{II}} = \left(\frac{1}{2} - \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}}, \frac{1}{2} + \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}}, \frac{1}{2} - \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}}\right)$$

$$(x_{eq},\, y_{eq},\, z_{eq})_{\mathrm{III}} = \left(\frac{1}{2} + \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}}, \frac{1}{2} - \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}}, \frac{1}{2} + \frac{\sqrt{6\rho_2 - 1}}{2\sqrt{(2\rho_2 - 1)^3}}\right)$$

We can notice that these equilibria are exactly those of the majority-anti majority model for what concerns the variables x and y. The two models behave in a very similar way and indeed the stability analysis shows that a bifurcation occurs at $\rho_2 = \frac{1}{6}$ even in this case, with equilibria II and III that are unstable for $\rho_{<}\frac{1}{6}$, when equilibrium I is stable, and become stable for $\rho_2 > \frac{1}{6}$, when equilibrium I becomes unstable. This similarity is not that surprising if we think about the structure of the star graph: the voter-antivoter agents interact with each other only indirectly through the central node C, which follows a 3-majority dynamic.

In the following, we show some trajectories of system (142) projected onto the x, y plane, as well as the corresponding sample paths of the underlying Markov process with $q_c = 0.6$ and for different values of the parameter ρ_2.

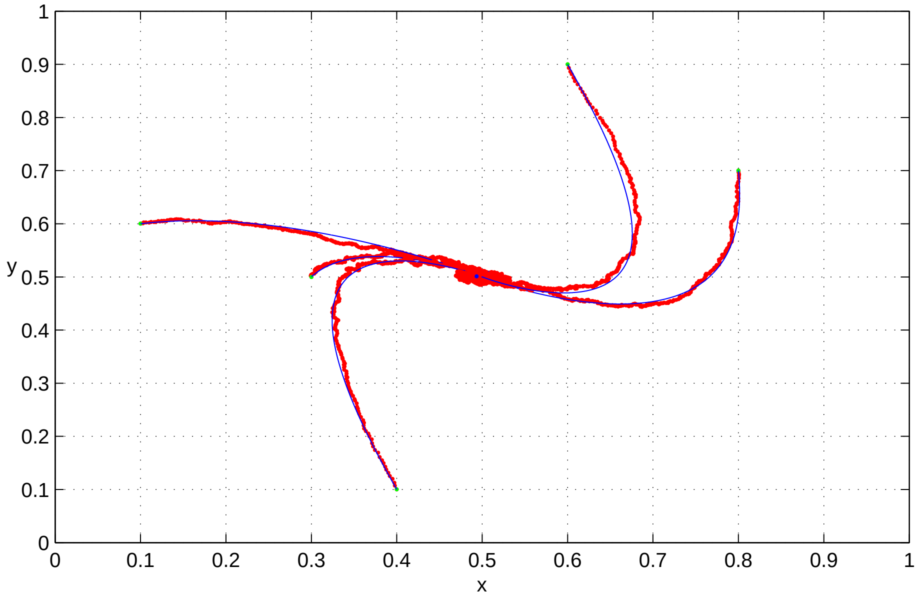


Figure 27: In blue, some trajectories of the model projected on the x, y plane. In red the corresponding sample paths of the markov chain, which considers $N = 10000$ agents, are shown. The values used are: $\rho_2 = 0.6$, $q_c = 0.6$, $q = 0.7$.

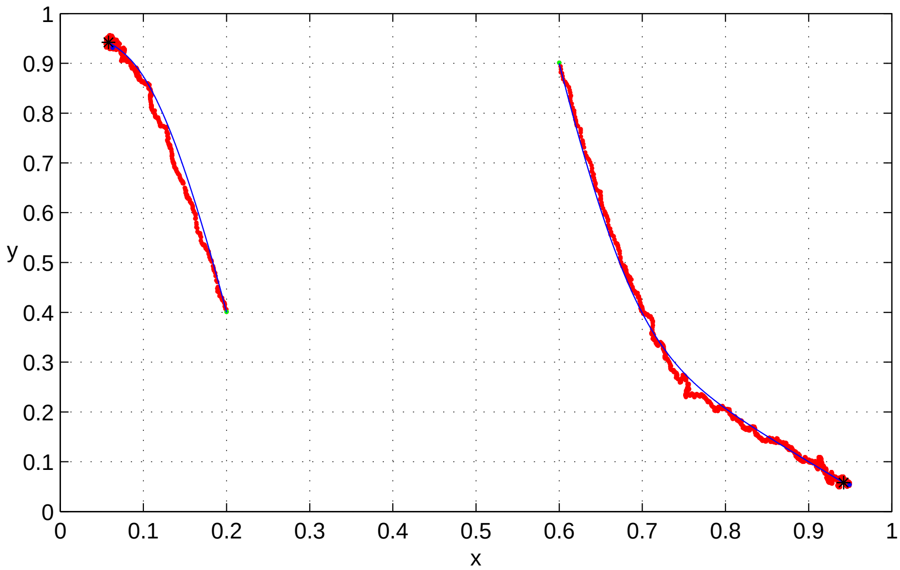


Figure 28: In blue, two trajectories of the model projected on the x, y plane. In red the corresponding sample paths of the markov chain, which considers $N = 10000$ agents, are shown. The values used are: $\rho_2 = 0.1$, $q_c = 0.6$, $q = 0.7$.

we can appreciate how the determinist system (142) well approximates the dynamics of the underlying Markov process.

The case: $q_c = \frac{1}{N}$

If we choose $q_c = \frac{1}{N}$, the behaviour of the model changes drastically. Indeed, in this case we pick up node C only once in a while (in a timespan of $t = k * N$ it will be choosen k times on avarage. Figure 29 shows a sample path of the markov chain for this value of q_c.

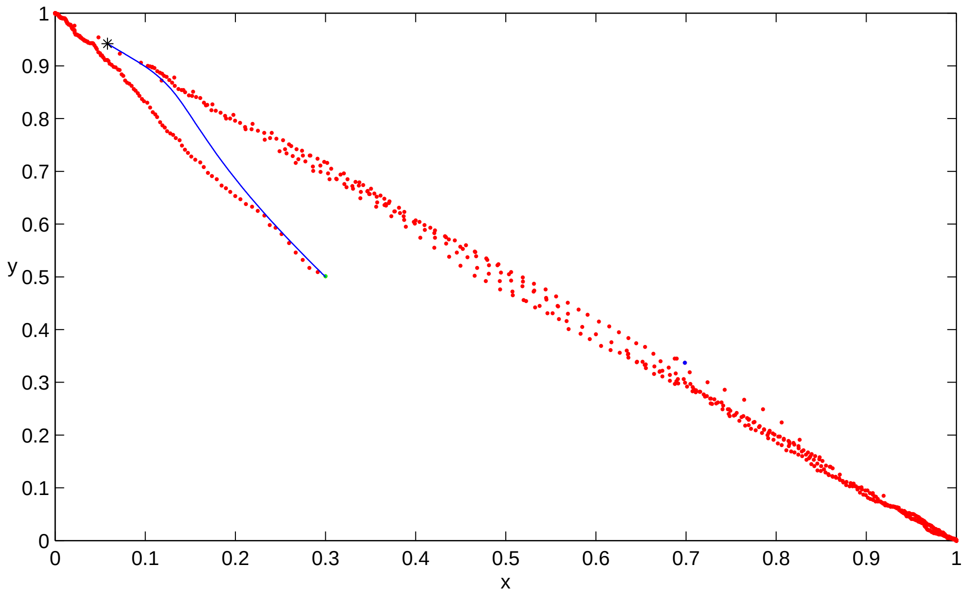


Figure 29: In blue it is plotted a trajectory of the model projected on the x, y plane, converging to equilibrium II; in red a sample path of the Markov chain starting from the same initial condition. The values used are: $\rho_2 = 0.1$, $q = 0.7$

From the previous figure we can see that the Markov chain behaves in a similar manner . From the previous figure we can observe that the system oscillates between the two polarization states and this happens independetly on the value of ρ_2. Such dynamics makes perfectly sense if we think that the central node C will be selected only once in a while and, conversely, the periferic nodes are chosen for an update most of the time. This means that, if we start with the central node in a certain state, say 1, then for a long time only the periferic nodes interact and the system will approach a polarization state (voter in state 1, antivoter in 0). After some time, the node C will be selected and with a certain probability it will flip its opinion; consequently, the system will go towards the other polarization state.

This kind of behaviour is not captured by the differential system (142). Indeed, in this case we cannot simply substitute the random variable $C(t)$ with its mean value at time t and a determinist description of the process doesn not hold anymore. Neverthless, one can still simulate the stochastic system (134), which holds for any value of q_c.

More in details, simulations can be easly performed by considering that the node C can actually change its status only when it has been choosen for an update at a certain time t. At each discrete time step, the node C is selected with probability $q_c = \frac{1}{N}$. Hence, one can define a bernoulli variable for each discrete time step A_k, such that it assumes value 1 with probability $\frac{1}{N}$ and value 0 with probability $1 - \frac{1}{N}$; these variables are all i.i.d. Thus, we can count the number of times that the node C is selected in a time t that scales with N. It is well know that when $N \to \infty$, this counting process converges to a continuous time Poisson process with rate $\lambda = 1$. Infact, we may notice that a unit of time t corresponds to N discrete time steps, thus, the number of times that the central node will be selected in such a unit of time is given by:

$$\sum_{j=1}^{N} A_j \sim B\left(N, \frac{1}{N}\right) \tag{143}$$

and then

$$B\left(N, \frac{1}{N}\right) \to Poiss(1) \qquad \text{for} \quad N \to \infty \tag{144}$$

One can then simulate first the epoch times of the Poisson process, which are the instant of times when the node C is chosen for an update. Then, it is sufficient to evaluate the probability that the node C assumes a certain status *given* that it has been chosen at a certain epoch time t_{ep}, that is

$$C(t)|_A = \begin{cases} 1 & \text{with probability} \quad m(t_{ep}) \\ 0 & \text{with probability} \quad 1 - m(t_{ep}) \end{cases} \tag{145}$$

In the figure 30 we show a simulation of system (134). A sample path of the underlying Markov chain is also plotted.

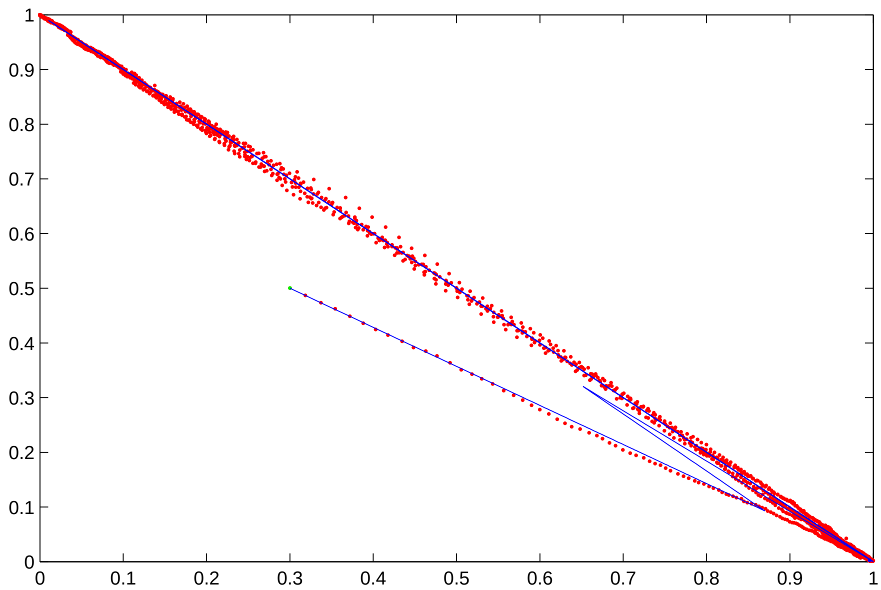


Figure 30: Dynamics of the model when $q_c = \frac{1}{N}$. The values used are: $\rho_2 = 0.1$, $N = 10000$, $q = 0.7, C(0) = 1$.

The intermediate cases

So far, we have investigated the dynamics of the model for two extreme cases: when q_c and $1 - q_c$ are of the same order and when $q_c = \frac{1}{N}$. It is interesting to investigate also the intermediate cases, when q_c ranges between these two values. In order to explore these scenarios, we run simulations for different values of q_c. In the following, we consider a star graph S_N where $N = 10000$.

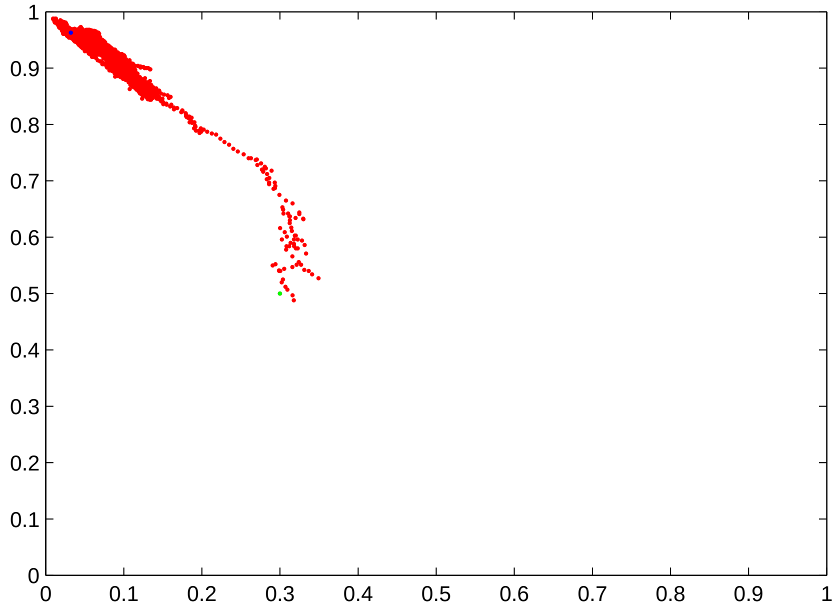


Figure 31: The values used are: $\rho_2 = 0.1$, $q_c = 1/\sqrt{N}$, $q = 0.7$.

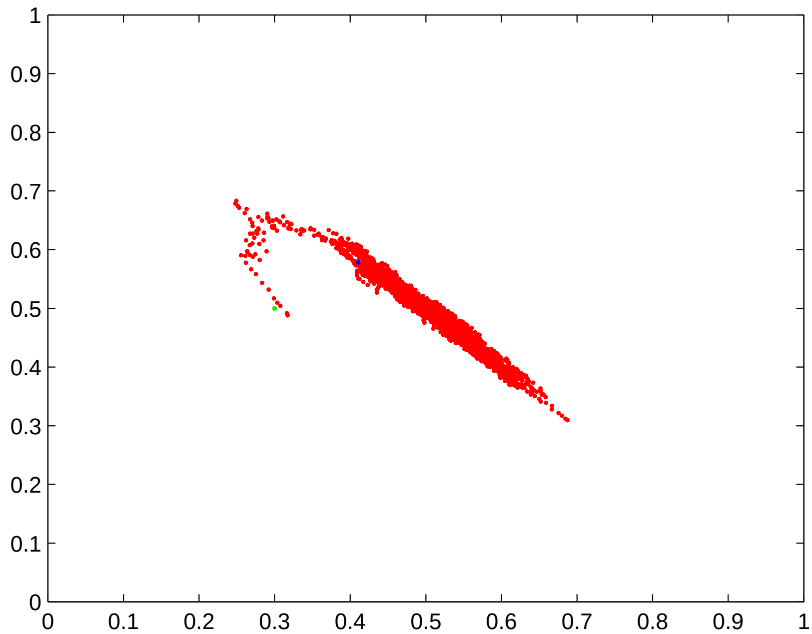


Figure 32: The values used are: $\rho_2 = 0.3$, $q_c = 1/\sqrt{N}$, $q = 0.7$.

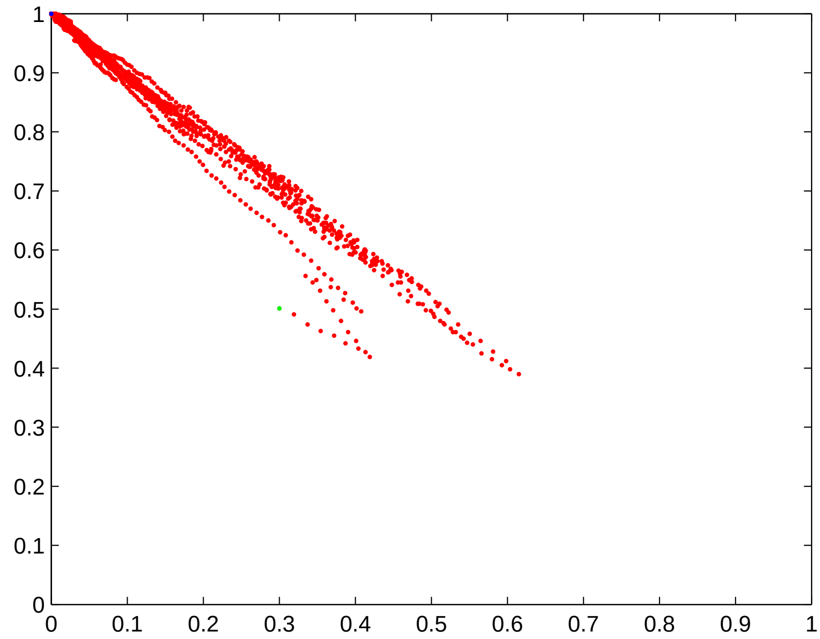


Figure 33: The values used are: $\rho_2 = 0.1,\ q_c = 0.001,\ q = 0.7.$

We can notice that the system faces a smooth phase transition as q_c ranges between the two limit cases. If we set $q_c = \frac{1}{\sqrt{N}}$ (figure 31 and 32), the sample paths follow a pattern that resembles the one obtained when $q_c \sim 1 - q_c$. Indeed, the system state is guided towards one of the stable equilibria of system (142), and it stays around these points but with much larger swings. Notice that we still have a bifurcation around $\rho_2 = \frac{1}{6}$. As q_c gets closer to the value $\frac{1}{N}$ (figure 33), the oscillations around the equilibria become larger and larger, and the system dynamic tends to be less deterministic, up to exhibit the oscillations between the two polarization points as shown in figure 30.

EPIOLOGUE

The purpose of this work was to design and study heterogeneous models for opinion dynamics. In the first part of this dissertation, we have described the stochastic approach to model opinion dynamics and we have highlighted the main features of it. At the same time, we have pointed out the importance of considering the fact that people exhibit a stunning diversity and react in different ways even if esposed to the same issues. In the second chapter, we have introduced the necessary mathematical background that we need in order to build and analyze heterogeneous model.

The second part of this work was devoted to designing heterogeneous models by considering two heterogeneous subpopulation interacting each other. Most of the time, we have considered the effect of anti social behaviours as perturbations affecting the normal route to consensus of dynamics like the voter and the majority laws. In particular, we have observed that several models of this kind exhibit phase transitions for certain thresholds regarding the size of the anti social subpopulation.

The analysis has been generalized by considering edge-based heterogeneous models in chapter three. In these kind of models populations have awareness of themselves, since they are able to recognize different people. In this context, we have shown a model of full majority that can be modelled by means of a differential equation with discontinuous right-hand side. Despite the fact that the Kurt's theorem does not hold for such a model, our simulations show that the differential system excellently approximates the dynamic of the underlying Markov chain and suggest that there is room to generalize the theorem in order to take into account even such irregular situations. Finally, we have designed a last model for which we dropped the mean field assumption by modeling an anti social behaviour on a star graph. A quasi-mean field description in terms of differential equations have been developed for suitable values of a key parameter of the model.

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