TARGETING INTERVENTIONS ON NETWORK SYSTEMS

BY

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DECLARATION

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

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ABSTRACT

In this dissertation, we consider a novel adversarial perturbation/protection problem for a class of network equilibria models emerging from a variety of different fields such as continuous network games, production networks, and opinion dynamic models. Specifically, we consider min-max problems whereby an external planner (the defender) aims at selecting the optimal network intervention within her given budget constraint in order to minimize a system performance that an adversary (the attacker) is instead trying to maximize. Problems are analytically solved for three particular cases of aggregate performances: the sum of squares of the equilibrium, the mean square or of its arithmetic mean. The main result is on the shape of the solutions, typically exhibiting a water-filling type structure with the optimal protection concentrated in a proper subset of the nodes, depending significantly on the aggregate performance considered. Our results also show that the optimal intervention of the defender has different regimes depending on the budget. For a large enough budget, the optimal intervention of the external planner acts on all nodes proportionally to new notions of network centrality. For lower budget values, such optimal intervention has a more delicate structure and is rather concentrated on a few target individuals.

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INTRODUCTION

1.1 INTRODUCTION AND MOTIVATION

The study of the effects of exogenous perturbations entering into a system, propagating and affecting its performance, and designing policies to mitigate them is a longstanding problem in many contexts of applied mathematics and theoretical engineering.

The growing field of multi-agent systems has opened new perspectives in this direction bringing in, particularly in the case of financial and economic models, the idea of *idiosyncratic shocks* [6, 23]. These are meant to be localized perturbations acting on the various agents of the system that aggregating through their interaction can possibly affect its macro evolution. The problem of understanding the effect of such perturbations in a system is ubiquitous, crossing all applications dealing with infrastructure systems (e.g. transportation networks, sensor and robotic networks [8, 22, 45, 55], electrical grids, financial and economic networks) but also of great interest in social network models [2, 4].

In the past decade, a conspicuous number of theoretical works in finance and economics (partly pushed by the recent financial crisis) [1, 3, 20, 35] have studied different instances of this problem, posing the fundamental questions of understanding the role of the network interconnections in propagating, aggregating and possibly amplifying exogenous perturbations of the system and of individuating the nodes most responsible for the spreading phenomenon. A key issue is that even if the shocks are originally independent along with the various agents of the system, their interaction will lead to correlations that will be the source of amplification effects. The extent of this phenomenon is expected to rely on the nature of the interactions and the topology of the network (e.g. the presence of highly connected nodes) [3, 5]. The concept of centrality, which scores the importance of various nodes, plays a crucial role in understanding the paired effect of shocks and network interactions [6, 26, 52]. In particular, it turns out that an economy is capable of reducing the effect of idiosyncratic shocks at a macroscopic level if and only if the network has no central hubs, i.e. nodes that are much more important than others [3].

Related to theoretical works in finance and economics, a large body of literature has focused on the impact of the social network structure on shaping the emer-

gent opinion profile in society [43]. The possibility of interfering in the democratic life of society in order to change the outcome of a political campaign or an election, alter the balance of power on a social issue, and legitimate or delegitimate a position is an unsettling concrete reality. Automatic programs, often referred to as "bots", are increasingly used to manipulate debates in social networks, spread fake news, and have become more sophisticated and harder to distinguish from real people. Phenomena such as *consensus, polarization*, or *persistent disagreement* have been studied in this context, as well as the role of targeted interventions in shifting individuals' opinions in the desired direction. Building on the Friedkin-Johnsen [34] opinion dynamic model, an extension of the French-De Groot [28] linear averaging model that admits exogenous inputs (e.g., stubborn nodes), various works in the recent past have studied the interaction between the network structure and the effect of external influences and formulated targeting optimization problems. In [56, 54], the problem of optimizing the position of a stubborn agent to maximally offset the effect of other influencing agents has been investigated. A corresponding adversarial problem formulated as a zero-sum game has been proposed and analyzed in [30]. In all these target intervention problems, the analysis leads to a specific concept of centrality: the most central nodes are those where it is more convenient to exert the action.

Entering into the specific of this context, in this dissertation we focus on *targeting intervention problems*, that aim to determine the nodes in a network on which the effect of an external action is maximum. The recognition of the central role played by this typology of problems in the analysis of networks made them the object of study in a variety of fields encompassing social and economic contexts. In quadratic game theory, the work [7] coined the term *key players* to indicate the players whose removal would affect the most aggregate performance of the system. Furthermore, in the context of more general linear best reply games, a different intervention aimed at changing the payoff function of prescribed players has been discussed [29, 37, 51]. In the context of strategic pricing and market powers, the optimal pricing strategy of a monopolist among consumers embedded in a social network has been studied in [21, 11, 14, 38].

Here, we propose and analyze an intervention problem formulated as an adversarial min-max problem: the first player (the attacker) can manipulate exogenous inputs that affect equilibrium points of the system, while the second player (the defender) applies a defense action weakening the strength of action of the first one (which, in mathematical terms translates into an in increase of the cost of the attacker). Actions of the defender are driven by external constraints that allow only particular and focused interventions on a proper subset of agents.

Emerging from the analysis are novel network centrality measures that indicate the most influential inputs on which the defender should mainly intervene. We

will show that the solution for the defender is to invest its mitigation resources on as many nodes as possible according to the available budget: all of them if its sufficiently high, otherwise on a subset. The nature of perturbations, which the attacker manipulates, can further complicate the problem, making it much more challenging to understand the best subset of nodes to intervene on.

Along with the dissertation, we propose some relevant applications in which the adversarial problem is tailored to the specific features of the considered context.

1.2 ORGANIZATION OF THE DISSERTATION

The thesis is organized as follows.

- Chapter 2 introduces useful concepts of graph theory and the related notion of centrality, which are the basis for linear network systems and the adversarial min-max problems treated in this dissertation. After introducing some proper notations in Section 2.1, in Section 2.2, we present some basic notions of graph theory, together with some relevant results on nonnegative matrices. Then, in Section 2.3, we present the notion of network centrality, a fundamental tool in the analysis of targeting intervention problems.

- In Chapter 3, we consider linear network models and important applications of our results. Section 3.1 presents the benchmark linear network model that describes the overall system's functionality. The two main ingredients of the model are the network, described by a nonnegative square matrix, and the exogenous inputs vector, that describes how external sources could influence the network. Three performance measures related to equilibrium configurations are presented. Such relation appears in several socio-economic models, such as the Nash equilibria of an underlying game or the asymptotic configuration of a network dynamics model. In Section 3.2, we describe the Friedkin-Johnsen opinion dynamic model [34], a fundamental extension of the French-De Groot [28] linear averaging model, that admits the presence of exogenous inputs (e.g., stubborn nodes). Then, in Section 3.3 we present quadratic network games [18, 17, 44], where players strategically choose actions to maximize a quadratic utility function. Utility of each agent is a combination of standalone action and actions of her neighbors in the networks. We conclude the chapter with Section 3.4, describing production networks and Cobb-Douglas model [6, 23]. In the economic literature, this model has been widely used to describe the macroeconomic impact of idiosyncratic shocks, i.e. localized shocks at agents level [1, 3, 9, 35].

- Chapter 4 is devoted to present the adversarial min-max problem, the main topic of the dissertation. Considerable attention has been recently devoted, particularly in the economic literature, to the effects that a disturbance in the input vector can have on the network equilibrium, in particular, how perturbation at the level of single agents can possibly be amplified by the network interaction, and propagate to the other agents. We take a further step in this direction and consider a more complex model, where disturbances are complementary paired with protections, and we cast it into an adversarial min-max problem. Then, Section 4.2 presents the essential mathematical properties of both the optimization problem and the objective function. We conclude this chapter with the solution to the inner maximization problem in Section 4.3.

- In Chapter 5, we undertake the adversarial min-max optimization problem under specific assumptions on the nature of exogenous perturbations. Section 5.1 analyzes the minimization problem and shows its essential properties. We highlight the arising of different centrality measures related to performances. Then, Section 5.2 presents the optimal intervention that solves the aforementioned problem. Solutions to these optimization problems will typically exhibit a 'water-filling' structure with the optimal solution concentrated on a limited number of nodes. Section 5.3 is devoted to employing our main result in applicative contexts. We will show two critical applications: the minimization of aggregate volatility and the minimization in the Friedkin-Johnsen model when initial opinions are stochastic vectors.

- Chapter 6 presents the most important result of the dissertation. In this chapter, we undertake a fundamental study of the adversarial min-max problem removing any assumption on the matrix of exogenous inputs. In Section 6.1, we introduce the min-max problem stating important general properties and show that, depending on the performance measure, the problem changes considerably. In Section 6.2, we study and solve the adversarial problem for the first performance measure. We will show within this setting, the study of the external minimization problem allows the use of similar tools used in Section 5.2. Then, in Section 6.3, we study and solve the adversarial problem for the second performance measure. The minimization problem considered in this section is much more difficult respect to the previous case and could be defined as an eigenvalues optimization problem. The main contribution of this section is an explicit recursive solution of the min-max problem that shows how the optimal solution for the defender is to invest its mitigation resources on all nodes if her available budget is sufficiently high, or on just a subset of nodes otherwise. We end this chapter with Section 6.4 by present-

ing and studying the min-max problem for the third performance measure. Also in this case the minimization problem should be defined in the class of constrained eigenvalues optimization. This section shows a part of ongoing research and examine particular cases of the problem.

- We wrap up the thesis with Chapter 7 where we summarize the main contributions of this work as a whole in Section 7.1 and discuss possible extensions and open problems that are object of current and future research in Section 7.2.

BACKGROUND

This first chapter introduces useful concepts of graph theory and the related notion of centrality, which are the basis for understanding network systems in Chapter 3 and the solutions to adversarial problems in Chapter 5, and Chapter 6.

After introducing some proper notations in Section 2.1, in Section 2.2, we present some basic notions of graph theory, together with some relevant results on nonnegative matrices. Then, in Section 2.3, we present the notion of network centrality, a fundamental tool in the analysis of targeting intervention problems.

2.1 NOTATIONS

To begin with, we explain the basic notation to be used throughout this work. We view all vectors as column vectors and we use x' to denote the transpose of a vector $x \in \mathbb{R}^n$. The same holds for matrices. Given a vector $x \in \mathbb{R}^n$, $x_\mathcal{S}$ is the restriction of x on the set of indices $\mathcal{S} = \{1, 2, \ldots, n\}$. If $\mathcal{S} = \{i\}$, we simply write x_i. Given a matrix $M \in \mathbb{R}^{n \times m}$, $M_i \in \mathbb{R}^n$ denotes the i-th column of M. Similarly to vectors, $M_\mathcal{S}$ represents the sub-matrix obtained from M by selecting columns in the set $\mathcal{S}$. We denote with M_{ij} the (i, j) entry of matrix M. Given a vector $x \in \mathbb{R}^n$, $[x]$ means a diagonal matrix having x on the components of x on the main diagonal. We indicate with $\mathbb{1}$ the all-1 vector and with I the identity matrix, regardless of their dimensions.

The symbols $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the nonnegative and positive reals, respectively. The symbols $\mathbb{S}^+$ and $\mathbb{D}^+$ denote the set of symmetric positive semi-definite matrices and the set of nonnegative diagonal matrices, respectively. The order of $M \in \mathbb{S}^+$ or $\in \mathbb{D}^+$ will be clear from the context. Given a vector $x \in \mathbb{R}^n$, with $\|x\|$, we denote the Euclidean norm. The entry-wise partial order is considered on $\mathbb{R}^n$, so that, the inequality $x \leq y$ for $x, y \in \mathbb{R}^n$ means that $x_i \leq y_i$ for every $i = 1, 2, \ldots, n$. Throughout the dissertation, we consider the partial order defined by the convex cone of positive semi-definite matrices: given two symmetric matrices $M, K \in \mathbb{R}^{n \times n}$, we say that $M \preceq K$ if $K - M$ is positive semi-definite. With $M \leq K$ we indicate that matrix K is entry-wise greater than M, i.e. $M_{ij} \leq K_{ij}$ for every pair $i, j = 1, 2, \ldots, n$. Additional notations will be introduced throughout this work and explained when needed.

2.2 NOTIONS OF GRAPH THEORY

A *(weighted directed) graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is a mathematical entity identified by an ordered triple of objects:

- a set of $N \in \mathbb{N}$ *nodes*, usually labeled by positive integers numbers, grouped in the node set $\mathcal{V} = \{1, 2, \ldots, N\}$;

- a set of ordered pairs of nodes (i, j) with $i, j \in \mathcal{V}$, named *edges*, which are collected in the edge set $\mathcal{E} = \mathcal{V} \times \mathcal{V}$;

- a *weight matrix* $W \in \mathbb{R}_+^{\mathcal{V} \times \mathcal{V}}$ that has the property that $W_{ij} > 0$ if and only if there exists the edge from node i to node j, i.e. $W_{ij} > 0 \iff (i, j) \in \mathcal{E}$. This also means that we can associate a weighted directed graph $\mathcal{G}_W$ to any square matrix $W \in \mathbb{R}_+^{N \times N}$ with node set $\mathcal{V} = \{1, 2, \ldots, N\}$, edge set $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : W_{ij} > 0\}$ and weights defined by W.

The presence of a specific edge (i, j) has to be interpreted as a direct connection from node i to node j and the associated weight W_{ij} measures the strength of the connection. Depending on the context, the presence of edge (i, j) may have different interpretations: it may indicate that node i influences node j, or, conversely, that i observes j, in the sense that i has access to the state of node j and get influenced by it. We shall refer to edges (i, i) whose head node coincides with its tail node as *self-loops*.

In certain applications, edges have an intrinsic bilateral meaning (e.g. symmetric interactions, human relations, partnership). This corresponds to the situation where the two edges (i, j) and (j, i) either belong to $\mathcal{E}$ and have the same weight $W_{ij} = W_{ji} > 0$, or are both absent. Graphs with this property are called *undirected*. We notice that if $\mathcal{G}$ is undirected, then its weight matrix $W = W'$ is symmetric.

If $W_{ij} \in \{0, 1\}$ for all $i, j \in \mathcal{V}$, the graph is called *unweighted*. The matrix W is called, in this case, *adjacency matrix*. If $\mathcal{G}$ is unweighted, it is often described by the pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. If a graph $\mathcal{G}$ is unweighted, undirected, and does not contain self-loops, i.e. $W_{ii} = 0$, it is called *simple*.

We introduce other useful notions related to the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$.

- The *out-neighborhood* and the *in-neighborhood* of a node $i \in \mathcal{V}$ are, respectively, the sets

$$\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}, \qquad \mathcal{N}_i^- = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}.$$

Nodes in $\mathcal{N}_i$ and $\mathcal{N}_i^-$ are referred to, respectively, as out-neighbors and in-neighbors of node i in $\mathcal{G}$. Nodes with no out-neighbors other than possibly themselves are called *sinks*, while nodes with no in-neighbors other than possibly themselves are called *sources*.

- The *out-degree* and *in-degree* of a node $i \in \mathcal{V}$ are defined, respectively, as

$$w_i = \sum_{j \in \mathcal{N}_i} W_{ij}, \qquad w_i^- = \sum_{j \in \mathcal{N}_i^-} W_{ji}.$$

Often we will use the shorter term degree for out-degree and the compact notation

$$w = W\mathbb{1}, \qquad w^- = W'\mathbb{1}.$$

- $\mathcal{G}$ is called *regular* if all its nodes have the same degree, i.e., if $w = w^- = \frac{1}{n}$.

Notice that in undirected graphs, there is no distinction between out- and in-neighbors, out- and in-neighborhoods, and out- and in-degree.

We conclude this section by presenting some relevant examples of graphs we will use in the rest of this dissertation. When referring generically to a graph, we will implicitly intend it to be weighted and directed, unless it is otherwise specified or clear from the context.

Example. (Complete graph). A complete graph K_N is a simple graph of N nodes each connected to every other node. We notice that for all $i \in \mathcal{V}$, $\mathcal{N}_i = \mathcal{V} \backslash \{i\}$ and $d_i = N - 1$. Hence, complete graph is regular. An example of complete graph is presented in Figure 1-(a).

Example. (Path graph). A path graph P_N with N nodes is a simple graph where the edge set is defined as $\mathcal{E} = \{(i, i+1), (i+1, i), \ i = 1, \ldots, N-1\}$. Path graph is thus undirected but not regular. An example of path graph is presented in Figure 1-(b).

Example. (Ring graph). A ring graph C_N is a simple graph of N nodes all of which have degree 2. We notice that C_N is derived by a path graph adding the undirected edge that connects node 1 to node N. By definition, ring graph is regular. An example of ring graph is presented in Figure 1-(c).

Example. (Star graph). A star graph S_N with N nodes is a simple graph where the edge set is defined as $\mathcal{E} = \{(1, i), (i, 1), \ i = 2, \ldots, N-1\}$, where the index 1 represents the central node. Simple star graph is thus undirected but not regular: all nodes but node 1 have degree 1, while node 1 has degree $N - 1$. An example of star graph is presented in Figure 1-(d).

(a) Complete graph.   (b) Path graph.   (c) Ring graph.   (d) Star graph.
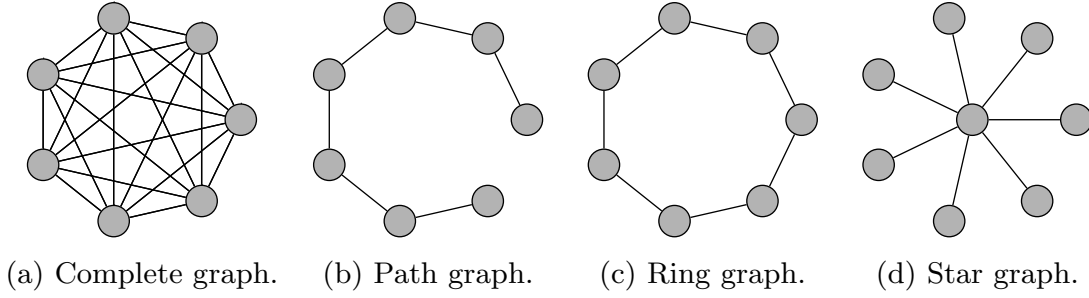
Figure 1: Relevant graph topologies presented in the examples of this section.

2.2.1 *Reachability and connected components*

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph. We introduce the following important definitions:

- A *walk* from node i to node j is a finite sequence of nodes $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ such that $\gamma_0 = i, \gamma_l = j$, and $(\gamma_{h-1}, \gamma_h) \in \mathcal{E}$ for all $h = 1, \dots, l$, i.e., there is a link between every two consecutive nodes. Here, l is called the length of the walk. By convention, we consider walks of length 0 as going from a node to itself;

- A walk $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ such that $\gamma_h \neq \gamma_k$ for all $0 \leq h < k \leq l$, except for possibly $\gamma_0 = \gamma_l$, is called a *path*. In plain words, a path is a walk that does not pass through a previously visited node except possibly for ending in its start node;

- A node j is said to be *reachable* from a node i if there exists a walk from i to j;

- A graph $\mathcal{G}$ is called *strongly connected* if given any two nodes i and j, we have that i is reachable from j;

- Given a subset of nodes $\mathcal{S} \subseteq \mathcal{V}$, we say that $\mathcal{S}$ is *globally reachable* (in $\mathcal{G}$) if $\mathcal{S}$ is strongly connected and for every $j \notin \mathcal{S}$ there exists a walk in $\mathcal{G}$ from j to some $i \in \mathcal{S}$.

The analysis of the connectedness of a graph can be further refined by considering the so called *connected components* of $\mathcal{G}$ that are the maximal subsets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k$ of the node set $\mathcal{V}$ such that, for every two nodes i and j in the same component $\mathcal{V}_h$, there exists a path from i to j. In other words, that means that the sub-graph associated to such a component is strongly connected. Note that the size of a connected component may range from 1 (in case there exists a node i such that there exists no other node $j \neq i$ such that both j is reachable
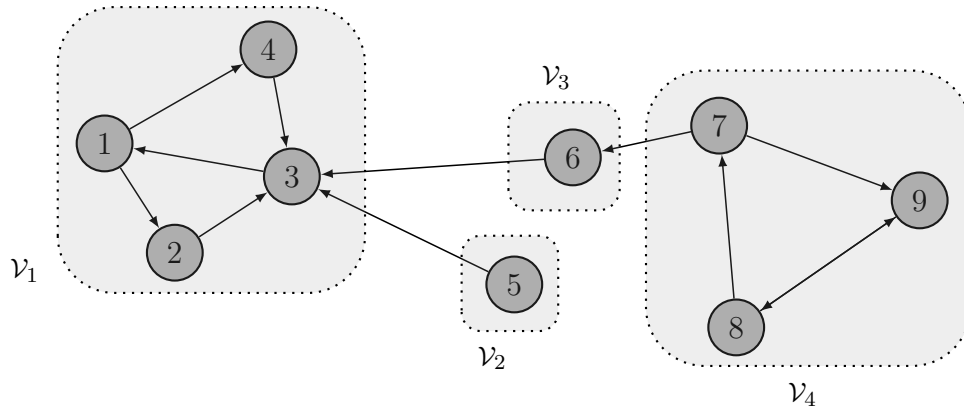
Figure 2: A graph with 10 nodes and 4 connected components. Notice that $\mathcal{V}_1$ is a globally reachable set.

from i and vice versa) to N (when the graph is strongly connected). The splitting in connected components constitutes a partition of the node set $\mathcal{V}$, i.e., one has that

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_k, \quad \mathcal{V}_h \cap \mathcal{V}_l = \emptyset, \quad h \neq l.$$

In Figure 2 we show a graph consisting of 9 nodes and 4 connected components. The connected component $\mathcal{V}_1$ is a globally reachable set and it is a *sink* in graph terminology. Intuitively, if we one starts moving at random from node to node according to the links present in the graph, it will eventually "fall" in $\mathcal{V}_1$ and unable to get back to any of the other components.

2.2.2 *Algebraic graph theory*

One of the key achievements of modern graph theory is the exploitation of the role of matrices and the fact that many graph properties admit an equivalent linear algebraic version. This section introduces some of these notions that will be useful later on.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, the first most natural matrix associated to it is of course the weight matrix W itself. The powers of W contain interesting information on the walks over $\mathcal{G}$, as shown below. First, let us define the *weight* of a length-l walk $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$ as the product of its l edge weights

$$W_\gamma = \prod_{1 \leq h \leq l} W_{\gamma_h \gamma_{h-1}},$$

with the convention that length-0 walks have unitary weights. The following property states that the (i, j)-th entry of the matrix power W^l coincides with the sum

of the weights of length-l walks from i to j and gives two important consequences of this fact.

Proposition 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph. Then, for every $l \geq 0$ and $i, j \in \mathcal{V}$,*

(i) $\left(W^l\right)_{ij} = \displaystyle\sum_{\substack{\gamma \ length-l \ walk \\ from \ i \ to \ j}} W_\gamma \,;$

(ii) $\left(W^l\right)_{ij} > 0$ *if and only if exists a walk of length l from i to j;*

(iii) $\mathcal{G}$ *is strongly connected if and only if for every $i, j \in \mathcal{V}$, there exists l such that $\left(W^l\right)_{ij} > 0$.*

In the special case of unweighted graph $\mathcal{G}$ all edges have unitary weight. Therefore, $\left(W^l\right)_{ij}$ coincides with the number of length-l walks from i to j.

Besides its weight/adjacency matrix, one matrix commonly associated to a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ turns out to be particularly relevant later on: the *normalized weighted matrix P*. In order to define P we shall assume that all nodes have positive out-degree, i.e. $w_i > 0$ for all $i \in \mathcal{V}$. This causes no real loss of generality since, if $w_i = 0$ for some node i, we can always modify $\mathcal{G}$ by adding a self-loop on i of some positive weight W_{ii}. Then,

$$P = [w]^{-1} W, \quad [w] = \mathrm{diag}(w). \tag{2.1}$$

Notice that all entries of P are non-negative: matrices with this property are simply referred to as non-negative. Moreover, by the definition of P it follows that

$$P\mathbb{1} \leq \mathbb{1}. \tag{2.2}$$

equation Nonnegative square matrices satisfying property (2.2) are referred to as *sub-stochastic* matrices. In plain words, a non-negative matrix is sub-stochastic if the sum of the entries in each row never exceeds 1.

Notice that in the literature it is often assumed that sub-stochastic matrices have the additional property that for at least one row there is strict inequality. Here we prefer not to follow this convention and in this way our class of sub-stochastic matrices contains also matrices P for which $w_i > 0$ for all $i \in \mathcal{V}$ and, hence, satisfying:

$$P\mathbb{1} = \mathbb{1}. \tag{2.3}$$

Nonnegative square matrices satisfying property (2.3) are also referred to as *stochastic matrices*.

In the following, we will denote the *spectral radius* of a matrix P, i.e., the largest absolute value of its eigenvalues, with the notation $\rho(P)$.

The structure of the normalized weight matrix P is also linked to the connectedness properties of the associated directed graph $\mathcal{G}_P$. In fact, a non-negative square matrix P is said to be *irreducible* if for every i and j, there exists $l \geq 1$ such that $\left(P^l\right)_{ij} > 0$. Equivalently, P is irreducible if and only if the associated graph $\mathcal{G}_P$ is strongly connected.

Finally, we present the theorem of Perron-Frobenius gathering known important results about non-negative matrices that can be found, e.g., in the monograph [10]. In the following, we will denote the *spectral radius* of a matrix P, i.e., the largest absolute value of its eigenvalues, with the notation $\rho(P)$.

Theorem 2.2.1 (Perron-Frobenius). *Let P in $\mathbb{R}_+^{N \times N}$ be a nonnegative square matrix. Then, the spectral radius $\rho(P)$ is an eigenvalue of P and there exists nonnegative vectors $x \neq 0$ and $y \neq 0$ such that*

(i) *$Px = \rho(P)x, \quad P'y = \rho(P)y$. Such vectors are called, respectively, a right and a left dominant eigenvector of P;*

(ii) *every eigenvalue μ of P is such that $|\mu| \leq \rho(P)$.*

Moreover, if P is irreducible or positive, then

(iii) *$\rho(P)$ is simple;*

(iv) *the dominant eigenvectors x and y are unique up to normalization and have all positive entries. y is also referred as to the eigenvector centrality of the graph and also as to the invariant probability vector of the graph (when it is normalized such that $\mathbb{1}'y = 1$).*

2.3 NETWORK CENTRALITY

We now want to study measures that capture the importance of a node's position in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. These are referred to as centrality measures, and vast literature exists on them [13]. We will not be exhaustive here but rather focus on a few key concepts naturally connected to the intervention problems we will study in the following chapters.

The simplest notion of centrality is the one of *degree centrality* whereby the importance of a node i is simply by its degree. Of course, in non-balanced networks one should decide whether the in-degree w_i^- (number of links pointing to node i) or the out-degree (number of links originating from node i) is to be used.

Measures of centrality differ depending on the types of statistics on which they are based, for example, degree centrality or closeness centrality. We will focus only on neighborhood centrality, which determines the importance of a node as

a function of its neighbors' importance. This concept goes beyond the number of neighbors and accounts for the fact that a node is more central if it is connected with other important nodes.

A natural extension of the (in-)degree centrality is the eigenvector centrality. The key idea is that connections from other nodes with high centrality should contribute more to the centrality of a node than connections from nodes with low centrality. Formally, we would like the centrality π_i of node i to be proportional to the sum of the centralities of the in-neighbors j of i, irrespective of their out-degree, i.e.

$$\pi_i \propto \sum_j \frac{W_{ji}}{w_j}\pi_j\,.$$

This corresponds to replacing the adjacency matrix W with its normalized version, the stochastic matrix $P = [w]^{-1}W$, and leads, considering that the dominant eigenvalue for P is 1, to the equation

$$\pi = P'\pi\,. \tag{2.4}$$

If $\mathcal{G}$ is strongly connected, matrix P results irreducible and therefore π results the nonnegative dominant left eigenvector of P. If we admit the normalization $\pi'\mathbb{1} = 1$, then π is unique and called the *eigenvector centrality* of $\mathcal{G}$.

Nodes can increase the eigenvector centrality of a given node arbitrarily by adding a self-loop on this node of a very large weight. In the limit, as the weight of a self-loop grows large, the ratio between the centrality of this node and the total centrality of all other nodes grows unbounded, even without losing connectivity. Even if self-loops are not allowed, one can easily take two nodes and add an undirected link between them of larger and larger weight: in the limit, as the weight of this undirected link grows large, the ratio between the sum of the two nodes' centralities and the centralities of all other nodes grows to infinity. This drawback is overcome by modifying the notion of centrality by allowing nodes to get some centrality, independently of their in-neighbors. Formally, let us choose some parameter $\lambda \in (0,1]$, and a nonnegative vector μ to be thought of as some intrinsic centrality. The standard choice is $\mu = \mathbb{1}$ so that all nodes have identical intrinsic centrality. Then, we can define the *Katz centrality* vector [46, 15] (also called PageRank centrality [40]) as the solution π of

$$\pi = \lambda P'\pi + (1-\lambda)\mu\,. \tag{2.5}$$

Observe that the dominant eigenvalue of $\lambda P'$ is equal to λ, so that $(I - \lambda P')$ is invertible. Using the expansion of the geometric series, the Katz centrality vector can be expressed as

$$\pi = (1-\lambda)\sum_{k\geq 0}\lambda^k(P')^k\mu = (1-\lambda)\mu + (1-\lambda)\lambda P'\mu + (1-\lambda)\lambda^2(P')^2\mu\,. \tag{2.6}$$

Equation (2.6) shows how the Katz centrality π_i of a node i can be expressed as a convex combination of its own a *priori* centrality μ_i and of the terms $(\mu'P')_i^k$ that depend on the centralities of the other nodes within distance k from i. The form of the corresponding weight coefficients $\lambda(1-\lambda)^k$ show that the way nodes influence the determination of the Katz centrality π_i of a node i, is exponentially decreasing with respect to the distance.

The next table helps to better figure out all the four centrality presented in this section.

in-degree	out-degree	eigenvector	Katz
$\pi = w^-$	$\pi = w$	$\pi = P'\pi$	$\pi = \lambda P'\pi + (1-\lambda)\mu$

Table 1: Resuming table of all centrality presented in this section: in-degree, out-degree, eigenvector (2.4), and Katz (2.5) centrality.

We conclude this section by presenting the calculation of Katz centrality for two classes of graphs presented in Section 2.2 and a small directed network. In order to keep the notation and the presentation simple, we shall assume that the intrinsic centrality vector μ is equal between nodes, i.e. $\mu = \mathbb{1}$. Thus, Katz centrality results

$$\pi = (1-\lambda)\left(I - \lambda P'\right)^{-1}\mathbb{1}.$$

Moreover, we assume the normalization constraint $\mathbb{1}'\pi = 1$.

Example. (Regular graphs). We recall that a simple regular graph satisfy $w = \alpha\mathbb{1}$, $\alpha > 0$. For example, K_N and C_N belong to the class of simple regular graphs. In this case, the matrix P results symmetric and therefore $P'\mathbb{1} = \mathbb{1}$. Matrices that satisfy the conditions $P\mathbb{1} = \mathbb{1}, P'\mathbb{1} = \mathbb{1}$ are referred to as *doubly-stochastic* matrices. Given that a matrix P associated to a simple regular graph is symmetric and stochastic, it results doubly-stochastic.

If P is doubly-stochastic, then also $(1-\lambda)\left(I - \lambda P'\right)^{-1}$ results doubly-stochastic. Therefore, if $\mathcal{G}$ is a simple regular graph it results that

$$\pi = N^{-1}\mathbb{1}.$$

Example. (Star graph). Consider the simple star graph S_N with N nodes. We denote with the subscript 1 the hub and with j any other marginal node. For symmetry reasons, Katz centrality π solves the following system of two equations

$$\begin{cases} \pi_1 - (N-1)\lambda\pi_j = (1-\lambda) \\ \pi_j - \lambda\pi_1/(N-1) = (1-\lambda). \end{cases}$$

Then, the normalized version of π results

$$\pi_1 = \frac{1 + (N-1)\lambda}{N(1+\lambda)}, \qquad \pi_j = \frac{(N-1) + \lambda}{N(N-1)(1+\lambda)}.$$

Keeping λ fixed and increasing the network size by augmenting the number of leafs, i.e. making $N \to +\infty$, Katz centrality results

$$\lim_{N \to +\infty} \pi_1 = \frac{\lambda}{(1+\lambda)}, \qquad \lim_{N \to +\infty} \pi_j = 0.$$

This result implies that the importance of leafs reduces to zero while the importance of the hub remains notable.
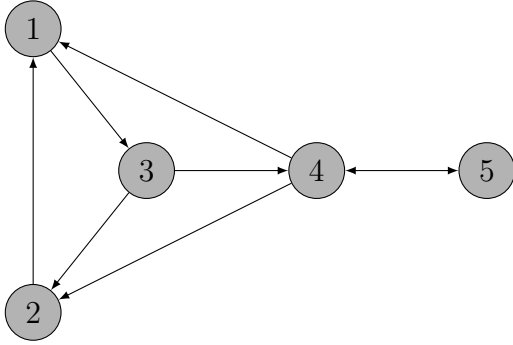
Figure 3: Directed network with $N = 5$ nodes.

i	w^-	w	π^e	π
1	**2**	1	**0.26**	**0.23**
2	**2**	1	0.2	0.19
3	1	1	**0.26**	0.21
4	**2**	**3**	0.2	0.22
5	1	1	0.06	0.14

Table 2: Centrality w^-, w, π^e, and π related to directed network of Figure 3.

Example. (Directed network). Consider the directed network of $N = 5$ nodes depicted in Figure 3.

In this illustrative example, we want to show the difference between the different centrality measures presented in this section. In particular, we compare in and out-degree, eigenvector centrality, and Katz centrality.

Starting from the adjacency matrix W, we calculate matrix P as defined in (2.1). Given that the graph is strongly connected, matrix P is irreducible so that we could calculate the unique eigenvector centrality. Setting $\lambda = 1/2$, we could then calculate the Katz centrality using (2.5). Columns of Table 2 show in-degree w^-, out-degree w, eigenvector centrality π^e, and Katz centrality π, respectively. Both centrality π^e and π are normalized to sum up to one. It is essential to notice that nodes' ranking is different depending on the centrality. For example, node 1 is the most important node concerning Katz centrality but is equally important to node 3 with respect to eigenvector centrality.

EQUILIBRIUM IN NETWORK SYSTEMS

This chapter presents the general linear network model that describes the benchmark of many applicative contexts. Then, we present the three most important applications of our study.

In Section 3.1, we start by presenting the linear network model that describes the overall system's functionality. Many different network models lead to equilibria configuration describable in form

$$x = (I - A)^{-1} Bu \, ,$$

where $A \in \mathbb{R}^{n \times n}$ is a matrix having spectral radius less than 1, $B \in \mathbb{R}_+^{n \times m}$ is a nonnegative matrix, $u \in \mathbb{R}^m$ is a vector of inputs that can be exposed to external perturbations, and the vector $x \in \mathbb{R}^n$ is the system equilibrium configuration. This is the reduced-form representation of various structural models and appear in several socio-economic models, such as the Nash equilibria of an underlying game or the asymptotic configuration of a network dynamics model. Then, we introduce quadratic performance measures of the equilibria configurations that evaluate the system as a whole.

In Section 3.2 we describe the Friedkin-Johnsen opinion dynamic model [34], a fundamental extension of the French-De Groot linear averaging model [28], that admits the presence of exogenous inputs (e.g., stubborn nodes). In recent years, this model has been used to investigate phenomena such as polarization of social opinions, disagreement, and absolute displacement and their mitigation through interventions [25, 36, 39, 48].

Then in Section 3.3, we present quadratic network games [18, 17, 44]. In quadratic network games, players strategically choose actions to maximize a quadratic utility function. The utility of each agent is a combination of standalone action and actions of her neighbors in the networks. In most of the works, the presence of an external planner is considered. The goal of the external planner, which could or could not have full information, is to intervene on the network to optimize an aggregate performance, e.g., total welfare [12, 19, 29, 37, 38, 52].

We conclude this chapter with Section 3.4, describing production networks and Cobb-Douglas model [6, 23]. In the economic literature, this model has been widely used to describe the macroeconomic impact of idiosyncratic shocks, i.e. localized shocks at agents level [1, 3, 9, 35]. The main result of these works is that

aggregate volatility, a measure that describes fluctuations at an aggregate level, is strictly connected to network topology through the vector of Katz centrality.

3.1 NETWORK EQUILIBRIUM

We consider a set of n (regular) agents $\mathcal{R} = \{1, 2, \ldots, n\}$ interacting through a directed graph $\mathcal{G} = (\mathcal{R}, \mathcal{E})$. The strength of the interactions is determined by a matrix $A \in \mathbb{R}_+^{n \times n}$. We assume that $A_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$.

The system is also composed of a set of exogenous sources $\mathcal{S} = \{n + 1, n + 2, \ldots, n + m\}$ that inject an input value $u \in \mathbb{R}^m$. Each agent $i \in \mathcal{R}$ is stimulated by a subset of exogenous values $u_{\mathcal{S}'}, \mathcal{S}' \subseteq \mathcal{S}$. The strength of the influences of sources on regular agents is determined by a matrix $B \in \mathbb{R}_+^{n \times m}$.

The functionality of the overall system is described by a vector $x \in \mathbb{R}^n$ whose components describe the level of activity (state) of the regular agents and satisfies the following balance equation

$$x_i = \sum_{j \in \mathcal{R}} A_{ij} x_j + \sum_{s \in \mathcal{S}} B_{is} u_s, \quad i \in \mathcal{R}. \tag{3.1}$$

Here $A_{ij} \in \mathbb{R}_+$ expresses the strength of influence of agent j on agent i, $B_{is} > 0$ expresses the strength of influence of source s on agent i, and u_s is the value injected by source s. In this case, relation (3.1) states that the state of each agent is a linear combination of its neighbors state and of exogenous inputs, where the weights are given by the rows of the network matrix A and the values B. We may then interpret the matrix

$$G = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

as the weighted adjacency matrix of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with node set $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$ and link set $\mathcal{E} = \{(i, j) \mid G_{ij} > 0\}$. We notice that, in this graph, sources have no outgoing links (they are sinks in the graph terminology). A link from a regular agent i to a source k indicates a direct positive influence exerted by the exogenous source k on agent i. Instead, a walk from a regular agent i to a source k indicates an indirect influence mediated by other regular agents.

Figure 4 shows a network composed by two regular agents $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2\}$ and two exogenous sources $\mathcal{S} = \{\mathcal{S}_3, \mathcal{S}_4\}$. We have represented regular agents with circle and exogenous sources with rectangle in order to highlight the difference between this two type of nodes. Exogenous values u_1 and u_2 enter into the system through sources $\mathcal{S}_3$ and $\mathcal{S}_4$ and propagate to regular nodes $\mathcal{R}_1$ and $\mathcal{R}_2$ with strengths described by the matrix $B \in \mathbb{R}^{2 \times 2}$.
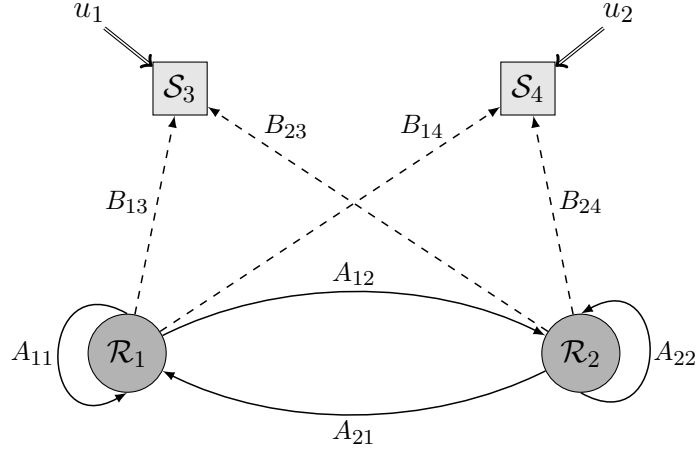
Figure 4: Example of a network composed by two regular agents $\mathcal{R} = \{1, 2\}$ and two exogenous sources $\mathcal{S} = \{3, 4\}$.

We shall see in following sections that relation (3.1) appears in different contexts: it describes the asymptotic opinion in the Friedkin-Johnsen opinion dynamics model, the Nash equilibrium of games with linear best-reply, or the output equilibrium in Cobb-Douglas production networks.

Throughout the dissertation we shall work under the following assumption.

Assumption 1. *The matrix A has spectral radius $\rho(A) < 1$.*

In all the applicative contexts presented in the following sections, the model described by (3.1) is studied under this assumption. This, in particular, implies that the solution to (3.1) exists and is unique.

The spectral assumption allows to rewrite the balance equation (3.1) as

$$x = Mu, \qquad M = (I - A)^{-1} B \,. \tag{3.2}$$

Such x is called the equilibrium configuration of the system. In many applications, it is natural to assume that the weights are normalized so that $\sum_j A_{ij} + \sum_s B_{is} = 1$ for every agent i. We notice that for a sub-stochastic matrix, the spectral radius condition can be easily checked at the graph topology level. Indeed, if we indicate with $\mathcal{R}_o$ the subset of agents i for which $\sum_j A_{ij} < 1$, we have that $\rho(A) < 1$ if and only if $\mathcal{R}_o$ is reachable from any node in the graph $\mathcal{G}$.

This condition is clearly satisfied when $\mathcal{R}_o = \mathcal{R}$, namely when exists $B_{is} > 0$ for every regular agent $i \in \mathcal{R}$ in the network. Notice that Assumption 1 allows one to express the matrix M as the limit of the matrix series

$$M = \sum_{k=0}^{+\infty} A^k B \,. \tag{3.3}$$

Observe that the series (3.3) admits a characterization in view of Proposition 1. Assuming $B = I$, A could be considered as the weighted matrix of a graph $\mathcal{G}_A$. Then, $M_{ij} = \sum_{k \geq 0}(A^k)_{ij}$ accounts for all the weighted walks from i to j of any length k, and describes the total influence that agents j has on agent i.

Significant attention has been recently devoted to how a modification in u can affect the solution x, described by (3.2), particularly in the economic literature [37, 38]. In some cases, this has been formalized as a *targeting intervention problem* where a planner looks for the optimal u (within a neighborhood of a reference value u^0) that maximizes some norm of x (e.g. total welfare). In other works, instead, u is considered as a *compensation*. The analysis aims at understanding its effects on the network equilibrium, in particular the role of the network topology in creating correlations and thus possible amplification of such shocks.

In line with past literature, we consider three basic performance measures formally presented as quadratic functions of the equilibrium state of the system x. Given that the system equilibrium depends on the exogenous inputs, we could represent these performance measures as function of $u \in \mathbb{R}^m$. Defining $J := n^{-1}\mathbb{1}\mathbb{1}'$, we now present the performance measures considered in this dissertation.

- The first measure is the square of the *average* of the system equilibrium denoted by Γ. This measure is a widely used performance to identify the central tendency of agents' state. By letting $\bar{x} := n^{-1}\mathbb{1}'x$, $\Gamma : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\Gamma(u) := (\bar{x})^2 = n^{-1}u'M'JMu. \tag{3.4}$$

  In some applications, x_i represents the action of agent i and $\mathbb{1}'x$ expresses the *total activity* of the system.

- The average of the agents' state does not capture all the information on the system. The second measure that we will consider in this dissertation is the *magnitude* of the system equilibrium, formally expressed as the norm of the vector x, defined by $\Phi : \mathbb{R}^n \to \mathbb{R}$ and defined as

$$\Phi(u) := \|x\|^2 = u'M'Mu. \tag{3.5}$$

- The last measure that we will analyze is the *global distance from the average* of the system equilibrium denoted by Ψ. This measures captures how agents' state are different respect to the average value $\bar{x}$. Formally, we define $\Psi : \mathbb{R}^n \to \mathbb{R}$ as

$$\Psi(u) := \|x - \mathbb{1}\bar{x}\|^2 = u'M'(I - J)Mu. \tag{3.6}$$

  In statistics, the global distance from the average correspond to the *population variance*, a measure that tells how data of a population are spread out.

The role of all these measures will be more clear in the following sections where we present applications in which they are used.

Remark. We observe that all the three performance measures (3.4), (3.5), and (3.6) could be presented in an unified way. Formally, for $Y \in \mathbb{R}^{n \times n}$, we define the function $F_Y : \mathbb{R}^m \to \mathbb{R}$ as

$$F_Y(u) = u' M' Y M u. \tag{3.7}$$

In particular:

- if $Y = n^{-1} J$, then $F_Y = \Gamma$;

- if $Y = I$, then $F_Y = \Phi$;

- if $Y = I - J$, then $F_Y = \Psi$.

We notice that, using properties of the trace, we can also write F_Y in the form

$$F_Y(u) = \mathrm{Tr}\left(u u' M' Y M\right). \tag{3.8}$$

In the following chapters we will mostly represents F_Y using this last formula.

Remark. These three performance measures are interlinked by the following conservation law

$$\Phi(u) = \Psi(u) + n\,\Gamma(u), \tag{3.9}$$

that expresses the magnitude of the system as the sum of an equating term, the average, and a dissociating term, the global distance from the average, among agents' state. Augmenting the magnitude keeping fixed the average results in an increase in the difference between agents' state.

The natural extension of this model is to consider a stochastic vector u. Particularly in the economic and financial applications, the random vector u models idiosyncratic shocks, i.e. shocks that are independent among agents. In other contexts, such as quadratic games, assuming a distribution on u helps to model uncertainty or incomplete information for an external planner that wishes to modify a system performance.

Assume u a stochastic vector with mean $\mathbb{E}[u] = 0\mathbb{1}$ and covariance matrix $\mathbb{E}[u u']$. In this dissertation, we also consider the expected value of the three performances presented before.

Remark. By the linearity of the expected value and the trace, we could write the conservation law of performance measures' expected value as

$$\mathbb{E}[\Phi(u)] = \mathbb{E}[\Psi(u)] + n\,\mathbb{E}[\Gamma(u)]. \tag{3.10}$$

This equivalence states that the *total variation* in the equilibrium state x, the expected value of Φ, is the sum of the variance of the average, the expected value of Γ, and the variance of the deviation from the average, the expected value of Ψ.

In the following sections, we present three interesting applications that exhibit the equilibrium configuration (3.2) and interpret the performance measures tailored to the specific context.

3.2 FRIEDKIN-JOHNSEN OPINION DYNAMICS

Consider a system made of a set of n agents $\mathcal{V} = \{1, 2, \ldots, n\}$ interacting through a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$.

The Friedkin-Johnsen model [34] in opinion dynamics can be described by

$$x(t+1) = [\lambda]Px(t) + (I - [\lambda])u, \tag{3.11}$$

where $P \in \mathbb{R}_+^{n \times n}$ is a stochastic nonnegative matrix representing interpersonal influences between agents and $[\lambda]$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \in [0, 1]$ for all i, is diagonal matrix of parameters that weight the role of network interactions. The vector $x(t) \in \mathbb{R}^n$ collects the opinion that a set of agents hold on certain fact at time t. In this context, input u_i is referred to as the *anchor* of agent i and is typically taken to be the original belief of the agent, namely $u = x(0)$. Notice that $[\lambda]_{ii} = \lambda_i$ indicates how much agent i is attached to its original belief and is thus resilient to the network interaction. Then, $x(t+1)$ results a convex combination of starting opinions and network effects.

If the set of nodes $\mathcal{S} = \{i \in \mathcal{V} \mid \lambda_i > 0\}$ is globally reachable, it can be shown that $[\lambda]P$ has spectral radius less than 1 and the Friedkin-Johnsen dynamic converges to the globally stable equilibrium x given by formula (3.2) with $A = [\lambda]P$ and $B = I - [\lambda]$ and u equal to the vector of initial opinions $x(0)$

Notice that, in contrast with the French-DeGroot model, although the asymptotic opinion is a convex combination of initial opinions $u = x(0)$, the consensus $\alpha\mathbb{1}, \alpha \geq 0$, is in general not reached.

In this context, it is interesting to consider performance measures that describes how far the asymptotic opinion (3.11) is from the consensus. Specifically, (3.5) coincides with *global displacement* $\|x\|^2$ and represents a measure of the distance from consensus configuration at 0. Whereas, (3.6) coincides with *polarization* $\|x - \bar{x}\mathbb{1}\|^2$ and measures how far the asymptotic configuration is from the average opinion $\bar{x}$. The conservation law (3.9) tells us that the global displacement is made by the polarization term and the average consensus of the asymptotic configuration, described by (3.4). Therefore, the problem of reducing the global displacement

should be addressed by shifting the average asymptotic opinion to 0 or by reducing difference in the final configuration, i.e. reducing the polarization.

Connected to the Friedkin-Johnsen dynamics there is also a learning interpretation of the model. Assume that $u = \bar{u} + \eta_i$, where $\bar{u} \in \mathbb{R}$ is the true value of a certain quantity and η_i are independent random variables with mean 0 and variances σ_i^2. Notice that, if no external perturbation is present, Friedkin-Johnsen dynamics leads convergence of all agents to the consensus value $x = \bar{u}\mathbb{1}$. In this context, $\mathbb{E}[\|x\|^2]$ represents a measure of the global displacement from such consensus configuration, while $\mathbb{E}[(n^{-1}\mathbb{1}'x)^2]$ measures how far the arithmetic mean of the asymptotic configuration is from $\bar{u}$. This last indicator is the crucial one to study how interaction may affect the so called wisdom of crowd. stable.

3.3 QUADRATIC GAMES

Consider a quadratic game among a set of players $\mathcal{V} = \{1, 2, \ldots, n\}$ where utilities have the form

$$\mathcal{U}_i(x) = u_i x_i - \frac{1}{2}x_i^2 + \beta \sum_{j \in \mathcal{V}} W_{ij} x_i x_j \,. \tag{3.12}$$

where the u_i's and β are positive constants and the elements W_{ij} are nonnegative. The benefits from increasing the action $x_i \in \mathbb{R}$ depends both on i's own action x_i and on others' actions. The first two terms of (3.12) give the benefits and the costs to player i of providing the action level x_i, respectively. The coefficients $u_i \in \mathbb{R}$ model the exogenous heterogeneity of agent i and are thus called *standalone marginal returns* and represents incentives of agent i in taking action x_i. The parameter $\beta > 0$ captures the facts that actions are *strategic complements*, i.e., each agents benefit from a positive action of other adjacents agents. The last term reflects the vantage of cooperation of i with his friends (those j for which $W_{ij} > 0$). Weights $W_{ij} \geq 0$ measure the strength of interaction between agent i and agent j. In general, it is assumed that interactions are symmetric, that is, $W_{ij} = W_{ji}$ for all $i, j \in \mathcal{V}$. Weights could be embedded in the matrix $W \in \mathbb{R}_+^{n \times n}$ that represents an undirected weighted network.

If the spectral radius of matrix βW is less than 1, the game has just one Nash equilibrium given by formula (3.2) with $A = \beta W$ and $B = I$.

There are several social measures that are useful in this case. *Social welfare*, that coincides with Φ, is defined as the sum of the equilibrium utilities $\|x\|^2$ and describes the wellness of a system. Consider, for example, a class of agents where u_i represents the incentives to study of agent i. Then, the total welfare reveals the total utility of the class, that is not only the sum of single players' incentives to study but also accounts for relations within the class. In this context of social activities,

the *social disparity* $\|(I - J)x\|^2$, described in our theory by Ψ, could be an useful index to detect difference in agents' actions at the equilibrium. It is important to notice that high social welfare could mean high social disparity and therefore both index should be taken into account.

If the network of agents describes relations between criminals, the average action has another interpretation [7]. The total activity $\mathbb{1}'x$ of a criminal network results proportional to the average action and, therefore, function (3.4) could be used to analyze it.

In the context of quadratic games it is natural to introduce an *external planner* that has the power to control agents incentives or the network to optimize the social performance of interest. In the example of the class of students, the planner could intervene to change incentives of agents in order to maximize the utility of the class [37]. In the example of the criminal network, the planner acts to remove a specific agent to minimize the total activity. However, if the planner has no full knowledge on the incentive values, it is natural to consider the vector u as a random vector and the expected value of the social performances.

3.4 PRODUCTION NETWORKS

Consider a static economy consisting of n competitive sectors denoted by $\mathcal{V} = \{1, 2, \ldots, n\}$, each of them producing a distinct product.

In the Cobb-Douglas model of an economy, firms in each sector transform intermediate inputs and labor into final products. Formally, the output y_i of firm i is given by

$$y_i = a_i e^{\omega_i} \ell_i^{\beta_i} \prod_{j=1}^{n} z_{ji}^{A_{ji}}, \quad i = 1, 2, \ldots, n, \tag{3.13}$$

where ω_i is the log-productivity shock, ℓ_i is the amount of labor employed by industry i, z_{ji} is the amount of good j used to produce good i, a_i is a constant, and β_i is the share of intermediate good in the production. The value $A_{ij} \geq 0$ indicates the share of good j in the production technology of good i.

In addition to firms, the economy is populated by a continuum of identical consumers, and each consumer is endowed with a unit of labor that firms can be hired for the purpose of production. The representative consumer has symmetric preferences among all the products of the economy given by

$$\mathcal{U}(c_1, c_2, \ldots, c_n) = \prod_{i}^{n} c_i^{\gamma_i}, \tag{3.14}$$

where c_i is the amount of good i consumed and γ_i is the consumer preference weight.

The competitive equilibrium of this economy consists of a collection of prices and quantities such that:

- for every representative firm $i = 1, 2, \ldots, n$ in each sector, the employed labor ℓ_i and the quantity of intermediate products $(z_{ij})_i$ maximizes its profits

$$\pi_i = p_i y_i - w \ell_i - \sum_j p_j z_{ji} \,,$$

  where p_i is the price of good i, while taking the prices and the wage w as given;

- the consumer vector c maximizes the utility $U(c_1, c_2, \ldots, c_n)$ given everything else;

- the market for goods and the market for labor clear, that is, for each good i must be satisfied

$$\sum_k \ell_k = 1 \,, \quad y_i = c_i + \sum_j z_{ij} \,, \quad i = 1, 2, \ldots, n \,.$$

The network topology of the economy is conveniently represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and where there is a directed link (i, j) from a node i to a node j if and only if $A_{ji} > 0$, i.e., if the output of node i is used as input in the production of node j. The effectiveness coefficients A_{ij} can be thought of as the entries of an $n \times n$ matrix A, which is also known as the *input-output* matrix of the network economy. We define $\alpha_i = \sum_j A_{ij}$ and assume the following normalizations:

$$\alpha_i + \beta_i \leq 1, \quad \sum_k \gamma_k = 1 \,.$$

The first inequality implies that A is a row-substochastic matrix (since $A_{ij} \geq 0$ and $\alpha_i = \sum_j A_{ij} \leq 1$ for every i), so that in particular its spectral radius $\rho(A) \leq 1$. Throughout, we shall assume that $\rho(A) < 1$.

Because of the assumption that $\rho(A) < 1$ the *Leontief matrix* $M = (I - A)^{-1}$ is well defined and can be expressed as the limit of the geometric series

$$M = \sum_{k=0}^{+\infty} A^k = I + A + A^2 + A^3 + \ldots \,.$$

It is possible to prove [3, 23, 6, 24] that, at the (Walrasian) equilibrium, the log-productions satisfies

$$\log y = M\omega + \kappa \,, \tag{3.15}$$

where κ is a constant term. Notice that, this equilibrium could be written in term of (3.2), assuming $x = \log y$, $B = I$, and $u = \omega$.

Within this context, a key role is played by the *influence vector* defined as

$$v := M'\mathbb{1}.$$

The main statistic that is used in economic and production literature is the *aggregate output*, defined as the log of the real value added in the economy $\mathbb{1}'x$.

The interconnections between different sectors may function as a propagation mechanism of productivity shocks throughout the economy. A crucial measure of economic fluctuations at an aggregate level is the standard deviation, or equivalently the variance, of the economy's aggregate output, which we refer to as the *aggregate volatility* of the economy. Assuming that shocks among firms are independent with mean 0 and variance σ_i^2, the aggregate volatility coincides with Γ. This formulation of the aggregate volatility highlights that fluctuations at the aggregate levels are linked to the structure of the economic network through the influence vector.

MIN-MAX ADVERSARIAL PROBLEMS ON NETWORK

In this chapter, we present the adversarial problem that will be the core of the dissertation.

In Section 4.1, we introduce the adversarial min-max optimization problem. Considerable attention has been recently devoted, particularly in the economic literature, to the effects that a disturbance in the vector u can have on the network equilibrium, in particular, how perturbation at the level of single agents can possibly be amplified by the network interaction, and propagate to the other agents. We take a further step in this direction and consider a more complex model, where perturbations are complementary paired with protections, and we cast it into an adversarial min-max problem. The first player (the attacker) can manipulate exogenous perturbations that influence agents, while the second player (the defender) applies a defense action weakening the action of the first one (which, in mathematical terms, translates into an increase in the cost of the attacker). The list of examples continues by drawing connections from the considered optimization problem and applications.

Then, Section 4.2 presents the essential mathematical properties of the optimization problem and the objective function. Formally, we will show that the objective function is a convex-concave function with respect to protections and perturbations and motivates our min-max approach. In Section 4.3, we conclude this chapter by providing the solution to the inner maximization problem, i.e., providing the optimal perturbation of the attacker.

4.1 THE ADVERSARIAL MIN-MAX PROBLEM

As presented in Chapter 3, we here consider linear models in the form

$$x = (I - A)^{-1} Bu \, ,$$

with the standing assumption that $A \in \mathbb{R}^{n \times n}$ has a spectral radius less than 1 and $B \in \mathbb{R}^{n \times m}$ is nonnegative matrix, and measure the performance of the equilibrium as described in Section 3.1.

In our research, we introduce a novel approach to study counter-actions at the nodes level to reduce the effects of a perturbation on u. Specifically, we assume that the vector u has the following structure:

$$u = \bar{u} + [\nu]^{-1}\omega \,. \tag{4.1}$$

where $\bar{u}_i$ is a reference value, ω_i is a variable modeling the perturbation and $\nu_i > 0$ is the protection actuated on node i. Below, we make precise our assumptions on the various terms.

- $\bar{u}$ is a given reference vector. In our analysis, the value of $\bar{u}$ does not play any role, for simplicity, from now on we assume that $\bar{u} = 0$.

- $\omega \in \mathbb{R}^m$ is a vector modeling the *exogenous perturbations*. Throughout the dissertation we will assume two kind of perturbations: deterministic (called *disturbances*) and stochastic (called *shocks*). If ω is a shock, we denote with K its covariance matrix, i.e., $\mathbb{E}[\omega\omega'] = K$. If ω is a perturbation, we denote with K the rank-one matrix generated by ω, i.e., $K = \omega\omega'$. We make the assumption that matrix K is bounded by 1 in trace and we consider the set of feasible perturbations, defined in terms of K, as

$$\Omega = \left\{ K \in \mathbb{R}^{m \times m} \mid K = K', \ K \succeq 0, \ \mathrm{Tr}(K) = 1 \right\} \,. \tag{4.2}$$

- $[\nu] = \mathrm{diag}(\nu_1, \nu_2, \ldots, \nu_m)$ where ν is the *intervention* vector. Considering a lower bound vector $d \in \mathbb{R}^m$, we make the assumption that $\nu_i \geq d_i$ for all i and we consider, given a constant budget $c \in \mathbb{R}$, such that $c \geq \|d\|^2$, the set of feasible intervention vectors defined as

$$\mathcal{Q}_c = \left\{ \nu \in \mathbb{R}^m \mid d_i \leq \nu_i, \ \sum_i \nu_i^2 \leq c \right\} \tag{4.3}$$

that is the protection vectors whose aggregated dislocations with respect to d is bounded by c.

We are now ready to present the adversarial min-max optimization problem, whose interpretation will be clarified in a few lines.

Let $A \in \mathbb{R}_+^{n \times n}$ and $B \in \mathbb{R}_+^{n \times m}$ such that $\rho(A) < 1$ and let $M = (I - A)^{-1}B$. For $Y \in \mathbb{R}^{m \times m}$ such that $Y = Y'$ and $Y \succeq 0$, we define the adversarial min-max optimization problem

$$\min_{\nu \in \mathcal{Q}_c} \max_{K \in \Omega} F_Y(\nu, K) \,, \tag{4.4}$$

where the objective function $F_Y : \mathcal{Q}_c \times \Omega \to \mathbb{R}$ is defined as

$$F_Y(\nu, K) = \mathrm{Tr}\left([\nu]^{-1} K [\nu]^{-1} M' Y M \right) \,. \tag{4.5}$$

We first observe that the above function derives directly from the quadratic performance (3.7). The interpretation is that of an adversarial model where two agents compete, an exogenous disturber that acts on K and a system defender that acts on ν. We now present in more detail each element of the problem.

The matrix $M \in \mathbb{R}^{n \times m}$ models the system formed by regular agents, whose interactions are described by matrix $A \in \mathbb{R}_+^{n \times n}$, and exogenous sources, whose influences on regular agents is described by matrix $B \in \mathbb{R}_+^{n \times m}$.

The matrix $Y \in \mathbb{R}^{m \times m}$ is related to performance measures of the system, presented in Section 3.1, and shapes the objective function of the adversarial min-max problem. Throughout the dissertation, we will consider three specific setting for matrix Y: the identity matrix I, the normalized all ones matrix J, and the centering matrix $I - J$, i.e., the matrix projecting on the space span$\{\mathbb{1}\}^\perp$, where span$\{v\}$ is the subspace spanned by a vector v.

The two sets $\mathcal{Q}_c$ and Ω model how the two adversaries can intervene on the system. Notice that the bound on the trace of K does not entail any loss of generality as a different bound can be absorbed in the bound of the defender. The defender has a limited budget c and moreover has a lower bound d on the single components. Typically, we will assume that $d_i \geq 1$ that means that the defender can not amplify the disturber action in any component.

Remark. In the case that ω is a random vector, we will consider the expected value of (3.4), (3.5), and (3.6). We observe that also the expected value of the three performance measures could be represented in the form (4.5). In fact, using the linearity of the expected value and letting $K = \mathbb{E}[\omega \omega']$, the following relation holds.

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Tr}\left(uu'M'YM\right)\right] &= \mathbb{E}\left[\mathrm{Tr}\left(\omega\omega'[\nu]^{-1}M'YM[\nu]^{-1}\right)\right] \\
&= \mathrm{Tr}\left(\mathbb{E}[\omega\omega'][\nu]^{-1}M'YM[\nu]^{-1}\right) \\
&= \mathrm{Tr}\left([\nu]^{-1}K[\nu]^{-1}M'YM\right) \\
&= F_Y(\nu, K).
\end{aligned}$$

We will deal with different scenarios restricting the set Ω, i.e., specifying the nature of the matrix of the exogenous input. In particular, throughout the dissertation, we will consider the subspace of diagonal nonnegative matrices and the entire space of symmetric positive semi-definite matrices. We now present each subspace in more detail.

The subspace of diagonal positive semi-definite matrices of size $m \times m$ with trace bounded by 1 will be denoted with $\mathbb{D}^+$ and is formally defined as

$$\mathbb{D}^+ := \left\{ K \in \mathbb{R}^{m \times m} : K = [\sigma^2], \sum_{i \in \mathcal{S}} \sigma_i^2 = 1 \right\}. \tag{4.6}$$

In the context of stochastic exogenous input, a matrix $[\sigma^2] \in \mathbb{D}^+$ characterizes the space of uncorrelated shocks, i.e., ω_i and ω_j, such that $\mathbb{E}[\omega_i \omega_j] = 0, \forall i \neq j \in \mathcal{S}$, and vector σ^2 represents the vector of variances.

The entire space of symmetric positive semi-definite matrices of size $m \times m$ with trace bounded by 1, already presented as Ω. We recall that a covariance matrix is always symmetric and positive semi-definite. Therefore, in the context of stochastic exogenous input, the set Ω exactly represents the set of all covariance matrices with bounded total variance.

We conclude this section by presenting a series of applications of the min-max optimization problem (4.4) and how we can write performances presented in Chapter 3 in the unified way expressed by (4.5). From now on, it is convenient to express performance $\Gamma, \Phi,$ and Ψ in terms of ν and K. Therefore, we will write accordingly $\Gamma(\nu, K), \Phi(\nu, K)$ and $\Psi(\nu, K)$.

Example. (Friedkin-Johnsen) Consider the Friedkin-Johnsen opinion model considered in Section 3.2.

Define $K = \omega\omega'$. The displacement $\|x\|^2$ coincides with $\Phi(\nu, K)$ and polarization coincides with $\Psi(\nu, K)$. The adversarial min-max problem (4.4) shall be then interpreted as follows.

The model for the input vector u in (4.1) can here be also interpreted as follows. Assume that ω is a random variable of mean 0 and variance σ_i^2 We refer to σ_i^{-2} as to the *expertise* of agent i. Assuming that the expertise depends on the single individual (e.g. school-level, access to information), we can interpret the multiplication of the disturbance ω_i's by the quantity $\nu_i \geq 1$ as an action aimed to increase the expertise of agent i to the new level $\nu_i^2 \sigma_i^{-2}$.

Example. (Quadratic games) Consider the quadratic games setting considered in Section 3.3.

In this context, the vector u represents the *standalone marginal return* or individual's incentives in taking action x_i First set $K = \omega\omega'$. Then the social welfare coincides with $\Phi(\nu, K)$, the social disparity coincides with $\Psi(\nu, K)$, and the squared total action coincides with $n^2 \Gamma(\nu, K)$

Assuming ω a random variable with mean $\mathbb{E}[\omega] = 0\mathbb{1}$ and covariance matrix $\mathbb{E}[\omega\omega'] = K$, we could consider the expected value of social performances accordingly.

In crime applications, where we seek for interventions to maximally diminish the activity of the system, it is natural to consider the minimization of the functional $\Gamma(\nu, K)$. Our proposed model studies the policy to minimize $\Gamma(\nu, K)$ within the intervention described in (4.1) and should be compared with the analysis in [7] where the authors study which player to remove in order to minimize the total activity.

Example. (Cobb-Douglas) Consider now the Cobb-Douglas model of economic networks considered in Section 3.4.

In this setting, ω_i represents the *log-productivity shock* of firm i and we assume $\mathbb{E}[\omega] = 0\mathbb{1}$ and $\mathbb{E}[\omega\omega'] = K$. Therefore, matrix K represents the covariance matrix of log-productivity shocks across economic sectors. Then, the aggregate volatility coincides with $\Gamma(\nu, K)$.

Our intervention strategy, that consists in adding the design parameter ν_i in the Cobb-Douglas model (3.13) such that it results

$$y_i = a_i e^{\omega_i/\nu_i} \ell_i^{\beta_i} \prod_{j=1}^n z_{ji}^{A_{ji}}, \quad i = 1, 2, \ldots, n,$$

leads to a corresponding modification of the covariance matrix K given by $[\nu]^{-1}K[\nu]^{-1}$. The optimization problem (4.4) we consider in this dissertation thus corresponds, in the context of production networks, to the minimization of volatility assuming an intervention policy that uses incentives to mitigate the effect of productivity shocks.

4.2 PROPERTIES OF THE OPTIMIZATION PROBLEM

In this section, we investigate the properties of problem (4.4). In particular, we will show that the outer min problem is convex in ν and the inner max problem is concave in K. This analysis motivates our approach of considering the min-max problem in place of the opposite max-min problem, showing that these two optimization problems are exactly the same.

As a first step, we reformulate the problem in order to highlight the aforementioned properties. In particular, we consider the change of variable $q_i = 1/\nu_i$, $i = 1, \ldots, m$. Considering $d \in \mathbb{R}^m_{++}$ and $c \geq \|d\|^2$, the feasible set of interventions $\mathcal{Q}_c$ becomes

$$\mathcal{O}_c = \left\{ q \in \mathbb{R}^m : 0 < q_i \leq 1/d_i, \ \sum_i q_i^{-2} \leq c \right\}. \tag{4.7}$$

Notice that, $\mathcal{O}_c$ is convex because it is the intersection of the convex function $\sum_i q_i^{-2} - c$ and the two halfspaces $q_i \geq \epsilon_i$ and $q_i \leq d_i$, where $\epsilon_i > 0$ is defined as follows. Consider the constraints $\sum_i q_i^{-2} \leq c, c \geq 1'd$ and fix all variables except i equal to their upper bounds, i.e. $q_j = 1/d_j$ for $j \neq i$. Then, it results that $q_i \geq \epsilon_i = \left(c - \sum_{j \neq i} d_j^2 \right)$. Using parameter ϵ we could rewrite the constraint $0 < q_i \leq 1/d_i$ as $\epsilon_i \leq q_i \leq 1/d_i$ removing the strict inequality.

The set Ω is convex and compact because it is the intersection of the closed convex cone of positive semi-definite matrices $\mathbb{S}^+$ with the affine hyperplane of equation $\mathrm{Tr}(K) = 1$.

Then, with the aforementioned change of variables, we could define the new objective function $G_Y : \mathcal{O}_c \times \Omega \to \mathbb{R}$

$$G_Y(q, K) := \mathrm{Tr}\left([q]K[q]M'YM\right), \tag{4.8}$$

and rewrite problem (4.4) in the new form

$$\min_{q \in \mathcal{O}_c} \max_{K \in \Omega} G_Y(q, K). \tag{4.9}$$

We now state an important property of (4.8).

Lemma 4.2.1. *The objective function $G_Y : \mathcal{O}_c \times \Omega \to \mathbb{R}$ is convex-concave, i.e. $G_Y(\cdot, K) : \mathcal{O}_c \to \mathbb{R}$ is convex in q for fixed K, and $G_Y(q, \cdot) : \Omega \to \mathbb{R}$ is concave in K for fixed q.*

Proof. Considering that the trace is a linear operator, $G_Y(q, \cdot)$ results linear in K for a fixed q, and, since it linear, it results also convex and concave. To prove convexity of $G_Y(\cdot, K)$ with respect to q for a fixed K, we could rewrite function G_Y as

$$G_Y(q, K) = q'\left(K \circ M'YM\right)q,$$

where $A \circ B$ denotes the Schur product of two matrices A and B of the same order, i.e., $(A \circ B)_{ij} = A_{ij}B_{ij}$. Given that both K and $M'YM$ are positive semi-definite also $(K \circ M'YM)$ results positive semi-definite [42]. Therefore $G_Y(\cdot, K)$ results a nonnegative quadratic form and hence a convex function of q. $\qquad\square$

The previous lemma ensures that the min-max problem is equivalent to the max-min problem. Formally, the following result holds true.

Proposition 2. *Consider $d \in \mathbb{R}_{++}$ and $c \geq \|d_i\|^2$. Then,*

$$\min_{q \in \mathcal{O}_c} \max_{K \in \Omega} G_Y(q, K) = \max_{K \in \Omega} \min_{q \in \mathcal{O}_c} G_Y(q, \Omega).$$

Proof. The result follows by Lemma 4.2.1 and by convexity and compactness of $\mathcal{O}_c$ and Ω. $\qquad\square$

This last result is the well known Minimax Theorem of Von-Neumann and Fan [32] and motivates our approach to the problem.

Given that problem (4.9) is convex in q for a fixed K and is equivalent to problem (4.4), also the latter results a convex problem in ν for a fixed K. Convexity of the problem also ensures that standard optimization techniques can be used to calculate the optimal protection numerically [16]. While many works have already presented practical and optimal methods to find the optimal solution, our research points to a theoretical analysis that aims at relating network topology to the nature of the optimal intervention.

4.3 WORST EXOGENOUS PERTURBATION

In this section, we investigate and solve the inner maximization problem

$$\max_{K \in \Omega} G_Y(q, K), \tag{4.10}$$

for the subset $\mathbb{D}^+$ and the entire space Ω. Given that G_Y is concave in K for a fixed $q \in \mathcal{O}_c$, we define the set of optimal points

$$K(q) = \underset{K \in \Omega}{\operatorname{argmax}} \ G_Y(q, K).$$

The next result shows the solution to problem (4.10).

Lemma 4.3.1. *Consider a fixed $q \in \mathcal{O}_c$.*

(i) If Ω is the entire space, then

$$\max_{K \in \Omega} G_Y(q, K) = \rho\left([q]M'YM[q]\right), \tag{4.11}$$

and the maximum is reached by any matrix $K \in \Omega$ such that

$$K = ww', \quad w \ s.t. \ , \ \|w\| = 1, \ [q]M'YM[q]w = \rho\left([q]M'YM[q]\right)w.$$

(ii) If $\Omega = \mathbb{D}^+$, then

$$\max_{K \in \mathbb{D}^+} G_Y(q, K) = \max_{i \in \mathcal{S}} M'_i Y M_i \ q_i^2, \tag{4.12}$$

and the maximum is reached by any matrix $K \in \mathbb{D}^+$ defined as

$$K = [w], \quad w \ s.t. \ , \sum_{i \in \mathcal{I}} w_i = 1, \quad \mathcal{I} := \{i \in \mathcal{S} : (M'_i J M_i \ q_i^2 = \max_j M'_j Y M_j \ q_j^2\}.$$

Proof. (i) For any $K \in \Omega$ consider the spectral decomposition $K = \sum_k \lambda_k w_k w'_k$ where w_k are the normalized eigenvectors of K (e.g. $w'_k w_k = 1$) and λ_k are the corresponding eigenvalues. We assume that eigenvalues are labeled in a decreasing order, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. We can compute $G_Y(q, K)$ as follows:

$$\begin{aligned}
G_Y(q, K) &= \operatorname{Tr}\left(\sum_{k=1}^m \lambda_k w_k w'_k [q]M'YM[q]\right) \\
&= \sum_{k=1}^m \lambda_k \operatorname{Tr}\left(w_k w'_k [q]M'JM[q]\right) \\
&= \sum_{k=1}^m \lambda_k w'_k [q]M'YM[q]w_k \\
&\leq \sum_{k=1}^m \lambda_k \rho\left([q]M'YM[q]\right) \\
&= \rho\left([q]M'YM[q]\right).
\end{aligned}$$

Let $\bar{w}$ indicate the eigenvector associated to the spectral radius ρ. Equality in the fourth line holds if and only if one of the w_k is a normalized version of $\bar{w}$, i.e., if $w_1 = \bar{w}/\|\bar{w}\|$, $\lambda_1 = 1$ and $\lambda_k = 0$ for every $k > 1$. This yields the result.

(ii) Consider $K \in \mathbb{D}^+$, defined as $K = [w]$, $w = (w_1, w_2, \dots, w_m)$. Function G_Y becomes the weighted sum

$$G_Y(\nu, K) = \sum_{k=1}^{m} w_k \left(M_k' Y M_k \, q_k^2 \right) .$$

Then, the problem is solved by any diagonal matrix $K = [w]$ concentrated on a subset of nodes $\mathcal{I}$ for which $(M'YM)_{ii} \, q_i^2 = M_i' Y M_i \, q_i^2$, $i \in \mathcal{I}$ results maximal. $\square$

Remark. Consider the entire space Ω. The optimal K is a rank-one matrix constructed using the eigenvector associated to the largest eigenvalue of matrix $[q]M'YM[q]$. Therefore, the deterministic case, i.e., when $K = \omega\omega'$, reaches the same maximum as the stochastic case, i.e., when $K = \mathbb{E}[\omega\omega']$. This result allows to embed the two instances in one single minimization problem.

In the same way we have defined $K(y)$, we could define $K(\nu)$, that is, the set of optimal points $K \in K(\nu)$ such that $F_J(\nu, K)$ is maximum. Notice that the results of Lemma 4.3.1 could be reformulated using ν_i^{-1} in the place of q_i, for all i.

In the following chapters, we will analyze the minimization problem

$$\min_{\nu \in \mathcal{Q}_c} F_Y(\nu, K), \quad K \in K(\nu),$$

for the different performance measures presented in Section 3.1 and for the two forms of $K(\nu)$ presented by Lemma 4.3.1. In particular, we will dedicate Chapter 5 to the case (4.12) and Chapter 6 to (4.11).

5

INDEPENDENT SHOCKS

In this chapter, we study the adversarial min-max under the assumption that matrix K belongs to the subset $\mathbb{D}^+ \subset \Omega$ of nonnegative diagonal matrices with bounded trace.

In Section 5.1, we analyze the minimization problem that arises from $K \in \mathbb{D}^+$. We will show the essential properties of (4.5) in the particular case of a diagonal matrix K and highlight the arising of different centrality measures related to performances.

Then, Section 5.2 presents the optimal intervention that solves the minimization problem of the defender. Solutions to these optimization problems will typically exhibit a 'water-filling' structure with the optimal solution ν concentrated on a limited number of nodes. The main message coming from our analysis is that the subset of nodes on which protection has to be taken to minimize the effect of perturbations is not only function of the network topology but also depends on the correlation pattern of the perturbations, as described by K, as well on the type of performance, we are considering.

Section 5.3 is devoted to employing our main result in applicative contexts. We will show two critical applications: the minimization of aggregate volatility and the minimization of opinion dynamics performances, e.g., polarization and disagreement, when initial inputs are stochastic vectors. Three main messages are coming from this analysis. The first one is that the optimal protection has to be proportional to the centrality related to the performance measure. In particular, the 'water-filling' structure suggests optimal protection has to level out inequalities among agents. The second goal is to highlight the difference between centrality measures that naturally arise considering different performances. In fact, centrality measures could considerably change the ranking of agents' importance. The last message is that parameters c and d highly influence the structure of the optimal protection. We will show how the budget c shapes the number of agents that could be protected and how heterogeneity in lower bound vector d could generate diverse protection even in regular networks.

Part of the work described in this chapter has been previously published in [26].

5.1 THE MODEL

In this section, we study the min-max optimization problem (4.4) in the special case when K is a nonnegative diagonal matrix. With an abuse of notation, we will indicate with $\sigma^2 = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2)$ the vector of diagonal elements of K, that is, $K = [\sigma^2]$. This analysis encompasses the case of independent shocks ω, having mean $\mathbb{E}[\omega_i] = 0\mathbb{1}$ and variance $\mathrm{Var}(\omega_i) = \sigma_i^2$. Given that shocks are independent, the covariance matrix $K = \mathbb{E}[\omega\omega']$ results a diagonal matrix having elements σ^2 on the main diagonal.

In this section we review some results of Chapter 4 applied to the specific context of diagonal matrix K. In the particular case of $K \in \mathbb{D}^+$ we obtain an easier form of the objective function (4.5).

Proposition 3. *Consider* $K = [\sigma^2] \in \mathbb{D}^+, \sigma^2 \in \mathbb{R}_+^m.$
Then,

$$F_Y(\nu, K) = \sum_{i \in \mathcal{S}} \left(\sigma_i^2 \, M_i' Y M_i \right) / \nu_i^2 \,. \tag{5.1}$$

Proof. Define $K = [\sigma^2]$, $\sigma^2 = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2)$. Equation (5.1) derives from (4.5) noticing that

$$\mathrm{Tr}\left([\nu]^{-1}[\sigma^2][\nu]^{-1}M'YM\right) = \mathrm{Tr}\left([\nu]^{-2}M'YM[\sigma^2]\right) \,.$$

The result follows by definition of trace. $\qquad\square$

It is important to notice that the i-th term $M_i'YM_i$ is nonnegative. In fact, given that $Y \succeq 0$, it results $M_i'YM_i \geq 0$. We could then define the *centrality index* π_i^2 equal to the i-th diagonal term of $M'YM$. Formally, we define

$$\pi_i^2 = M_i'YM_i, \quad i = 1, 2, \ldots, m \,, \tag{5.2}$$

that measures the importance of agent i respects to the performance Y considered. We now study the nature of π in the applications.

Example. (Friedkin-Johnsen) Consider the polarization index described by $\Psi(\nu, K)$. In line with [6], we define the *concentration centrality* of node i as

$$p_i = \|M_i - n^{-1}\mathbb{1}'M_i\| \,, \tag{5.3}$$

that defines how evenly agent i influence is distributed across the rest of the network. Then, centrality π results the concentration centrality, i.e. $\pi_i = p_i$. Notice

that this centrality index corresponds to the definition of population variance of vector M_i. Population variance is formally defined as

$$\delta^2(M_i) := \frac{1}{n} \sum_{j=1}^{n} M_{ji}^2 - \mu_i^2, \quad \mu_i = \frac{1}{n} \sum_{j=1}^{n} M_{ji}.$$

Then, it results that

$$p_i^2 = n\delta^2(M_i).$$

Example. (Quadratic games) Consider the expected value of social welfare described by $\Phi(\nu, K)$. In line with [52], we define the *cycle centrality* of node i as

$$\ell_i = \|M_i\|^2, \tag{5.4}$$

that describes the extent to which agent i is present in cycles of the network. Then, centrality π_i^2 coincides with cycle centrality, i.e. $\pi_i^2 = \ell_i$.

Example. (Cobb-Douglas) Consider the aggregate volatility described by $\Gamma(\nu, K)$. The well known Katz centrality, presented in Section 2.3 is here defined as

$$v_i = M_i'\mathbb{1}. \tag{5.5}$$

Then, centrality π_i is the Katz centrality, i.e. $\pi_i = v_i$

In the next section we present the most important result of this chapter, showing that the optimal intervention has to be proportional to the centrality π.

5.2 MAIN RESULT

In this section we present the solution of problem

$$\min_{\nu \in \mathcal{Q}_c} \max_{K \in \mathbb{D}^+} \sum_i (\sigma_i \, \pi_i / \, \nu_i)^2. \tag{5.6}$$

We will refer to π_i as to the *centrality* of agent i. We approach the min-max problem (5.6) by first analyzing the inner maximization problem. For convenience, to improve the readability of the results we present again the problem (4.10) in the particular case when $K \in \mathbb{D}^+$, that is, K is a nonnegative diagonal matrix. To this aim, we define the function $f_\pi : \mathcal{Q}_c \to \mathbb{R}$ as

$$f_\pi(\nu) = \max_{K \in \mathbb{D}^+} F_Y(\nu, K), \quad \nu \in \mathcal{Q}_c.$$

The next result, that has been already presented in Section 4.3, shows the solution of the inner maximization problem and the nature of function f_π in terms of the centrality vector π.

Lemma 5.2.1. *For every $\nu \in \mathcal{Q}_c$ it holds that*

$$f_\pi(\nu) = \max_{i \in \mathcal{S}} (\pi_i/\nu_i)^2, \tag{5.7}$$

and the maximum is reached by any matrix K defined as

$$K = [\sigma^2], \quad \sigma^2 \in \mathbb{R}^m \ s.t \ \sum_{i \in \mathcal{I}} \sigma_i^2 = 1, \quad \mathcal{I} := \{i \in \mathcal{S} : \ \pi_i/\nu_i^2 = \max_j \pi_j/\nu_j^2\}.$$

Proof. Proof follows by applying Lemma 4.3.1. $\qquad\qquad\qquad\qquad\qquad$ □

We are now ready to study

$$f_\pi(\nu^*(c)) := \min_{\nu \in \mathcal{Q}_c} f_\pi(\nu).$$

Without lack of generality, we assume that the elements of $\hat{\pi} = (\pi_1^2/d_1^2, \pi_2^2/d_2^2, \ldots, \pi_m^2/d_m^2)$ are ordered in decreasing order, i.e., $\hat{\pi}_1 \geq \hat{\pi}_2 \geq \cdots \geq \hat{\pi}_m$. From now on, when we refer to the i-th term of vectors d and π, we consider the order induced by $\hat{\pi}$. For example, the 2-nd term of d, indicated with d_2, is the value of d related to $\hat{\pi}_2$. Notice that the highest value of $\hat{\pi}$ is not necessary the node with the highest centrality π. In fact, the paired vector $\hat{\pi}$ embeds together centrality π and parameter d.

We now solve problem (5.6).

First we introduce the *water-level function* $T_{\pi,d}(\xi) : (0, +\infty) \to \mathbb{R}$ defined as

$$T_{\pi,d}(\xi) := \sum_{i=1}^{m} d_i^2 \max\{1, \hat{\pi}_i/\xi\}. \tag{5.8}$$

We notice that $T_{\pi,d}$ is continuous, strictly decreasing in $(0, \hat{\pi}_1]$, and

$$\lim_{\xi \to 0+} T_{\pi,d}(\xi) = +\infty, \quad T_{\pi,d}(\hat{\pi}_1) = \|d\|^2.$$

This implies that for every $c \geq \|d\|^2$, it is well defined

$$\xi(c) := T_{\pi,d}^{-1}(c).$$

Given that $T_{\pi,d}$ is strictly decreasing, then so is $\xi(c)$. An example of the water-level function is given in Figure 5.

We now define $k(c)$ as the maximum index such that $\hat{\pi}_{k(c)} > \xi(c)$. The following result holds true.
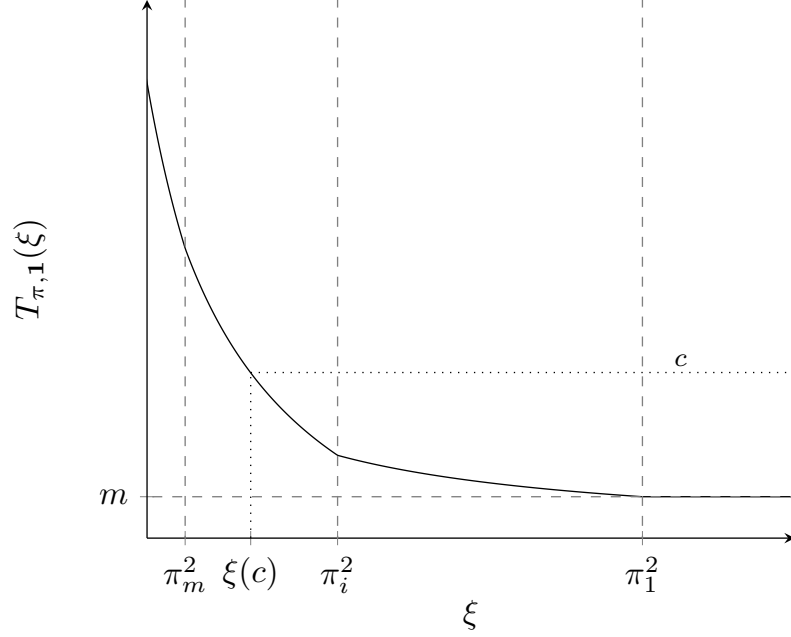
Figure 5: Water-level function $T_{\pi,d}(\xi)$ assuming $d_i = 1$ for all i. Vertical dashed lines indicate values where ξ equals π_i^2. The horizontal dotted line indicates a given value of c, while the vertical dotted line indicates $\xi(c) = T_{\pi,1}^{-1}(c)$.

Theorem 5.2.2. *Let $d \in \mathbb{R}_{++}$ and $c \geq \|d\|^2$.*
  It holds

$$\min_{\nu \in \mathcal{Q}_c} f_\pi(\nu) = \xi(c) \,, \tag{5.9}$$

and the optimum value for $\nu(c)$ is reached by

$$\nu_i^*(c) = \begin{cases} \pi_i/\sqrt{\xi(c)} & \text{if } i \leq k(c) \\ d_i & \text{otherwise} \,. \end{cases} \tag{5.10}$$

Proof. We can reformulate the starting introducing a new slack variable $\xi \in \mathbb{R}$ as

$$\min_{\xi \in \mathbb{R}, \nu \in \mathcal{Q}_c} \quad \xi$$
$$\text{s.t.} \quad (\pi_i/\nu_i)^2 \leq \xi, \quad i = 1, ..., m \,.$$

Define now the Lagrange multipliers $\mu \in \mathbb{R}_+, \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m, \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m$. The Lagrangian of the problem results

$$\mathcal{L}(\xi, \nu, \mu, \lambda, \alpha) = \xi + \sum_i \alpha_i \left((\pi_i/\nu_i)^2 - \xi\right) + \mu\left(\sum_i \nu_i^2 - c\right) - \sum_i \lambda_i(\nu_i - d_i).$$

Imposing first derivatives of $\mathcal{L}(\xi, \nu, \mu, \lambda, \alpha)$ respect to all variables equal to zero we obtain KKT conditions that are sufficient to find the optimal solution ν^*, ξ^* and optimal multipliers μ^* and α^*

$$\nu_i^* \geq d_i, \quad \mu^* \geq 0, \quad \alpha_i^* \geq 0, \quad i = 1, \ldots, m \tag{5.11}$$

$$\left(-2\alpha_i^* \pi_i^2 (\nu_i^*)^{-3} + 2\mu^* \nu_i^*\right)(\nu_i^* - d_i) = 0, \quad i = 1, \ldots, m, \tag{5.12}$$

$$\sum_i \alpha_i^* = 1, \quad \mu^* \left(\sum_i (\nu_i^*)^2 - c\right) = 0, \tag{5.13}$$

$$-2\alpha_i^* \pi_i^2 (\nu_i^*)^{-3} + 2\mu^* \nu_i^* \geq 0, \quad i = 1, \ldots, m, \tag{5.14}$$

$$(\pi_i / \nu_i^*)^2 \leq \xi^*, \quad \alpha_i^* \left((\pi_i / \nu_i^*)^2 - \xi^*\right) = 0, \quad i = 1, \ldots, m. \tag{5.15}$$

From now on we will omit the star * to indicate the optimal point and the optimal multiplier.

First notice that, if $\mu = 0$, then α_i must be equal to 0 to satisfy (5.14). Moreover, if $\mu = 0$, then also α_j for $j \neq i$ must be equal to 0 and therefore $\alpha_i = 0, \forall i$ and equation (5.13) is not satisfied because $\sum_i \alpha_i = 0 < 1$. Hence, $\mu > 0$ and $\alpha_i > 0$. Given that $\alpha_i > 0$, (5.15) imposes that

$$\nu_i^2 = \frac{\pi_i^2}{\xi}.$$

If $\xi > \hat{\pi}_i$ then ν_i results less than d_i and the system is not satisfied. Hence, $\xi \leq \hat{\pi}_i$ and we have to analyze two different cases.

Assume first that $\xi < \hat{\pi}_i$. Then, ν_i results greater than d_i and (5.12) also implies that

$$-\alpha_i \pi_i^2 / \nu_i^4 + \mu = 0.$$

If $\xi = \hat{\pi}_i$, then $\nu_i = d_i$.

Hence, we have obtained that the optimal solution could be written as

$$\nu_i = \max\left\{d_i, \pi_i / \sqrt{\xi}\right\}.$$

Plugging the optimal solution ν in the constraint $\sum_i \nu_i^2 = c$ we obtain

$$\sum_i \max\left\{d_i^2, \pi_i^2 / \xi\right\} = c.$$

Notice that the left hand side results equal to $T_{\pi,d}(\xi)$ by gathering d_i^2 for each term of the summation. Given that $T_{\pi,d}$ is invertible there exists an optimal ξ defined as $\xi = T_{\pi,d}^{-1}(c), \forall c > \|d\|^2$. Therefore, we could write optimal solution as

$$\nu_i = \max\left\{d_i, \pi_i / T_{\pi,d}^{-1}(c)\right\}.$$

□

We now comment on the results obtained. For simplicity of exposition, we assume that $d = \mathbb{1}$ in order to mainly focus on the role of centrality vector π. The vector π is then assumed ordered with a decreasing order. Then, the water-level function (5.8) results

$$T_{\pi,\mathbb{1}}(\xi) = \sum_{i=1}^{m} \max\left\{1, \pi_i^2/\xi\right\} .$$

The structure of the optimums we have found shows how the optimal protections are in general concentrated on a proper subset of nodes. In this respect, it is interesting to analyze various regimes depending on the chosen budget c.

1. Protection is active one just one node, namely $\nu_i^*(c) = 1$ for all $i > 2$, if and only if $\pi_2^2 < \xi(c)$ or, equivalently

    $$c < T_{\pi,1}(\pi_2^2) = (m-1) + (\pi_1/\pi_2)^2 . \tag{5.16}$$

    In this case, we get from (5.9) that the optimal value is given by

    $$f_\pi(\nu^*(c)) = (c - (m-1))^{-1} \pi_1^2 . \tag{5.17}$$

    We will refer to this as to the *low budget regime*. In this regime the node to be protected is node 1, that is, node with highest value of centrality π.

2. Protection is active on all nodes, namely $\nu_i^*(c) > 1$ for all i, if and only if $\pi_m^2 > \xi(c)$ or, equivalently,

    $$c > T_{\pi,1}(\pi_m^2) = (\|\pi\|/\pi_m)^2 \tag{5.18}$$

    We will refer to this as to the *high budget regime*. In this regime, we get from (5.9) that the optimal value is given by

    $$f_\pi(\nu^*(c)) = c^{-1}\|\pi\|^2 . \tag{5.19}$$

3. If $\pi_{i+1}^2 < \xi(c) < \pi_i^2$, then protection is active on a subset of nodes $\mathcal{U} = \{1, 2, \ldots, i\}$ and we get from (5.9) that the optimal value is given by

    $$f_\pi(\nu^*(c)) = \left(c - \sum_{j \notin \mathcal{U}} 1\right)^{-1} \sum_{i \in \mathcal{U}} \pi_i^2. \tag{5.20}$$

4. It follows from the shape of the optimum (5.10) that when the protection is active, i.e. $\nu_i^*(c) > 1$, then the level of protection $\nu_i^*(c)$ is proportional to the term π_i that resumes centrality but also the lower bound parameter.
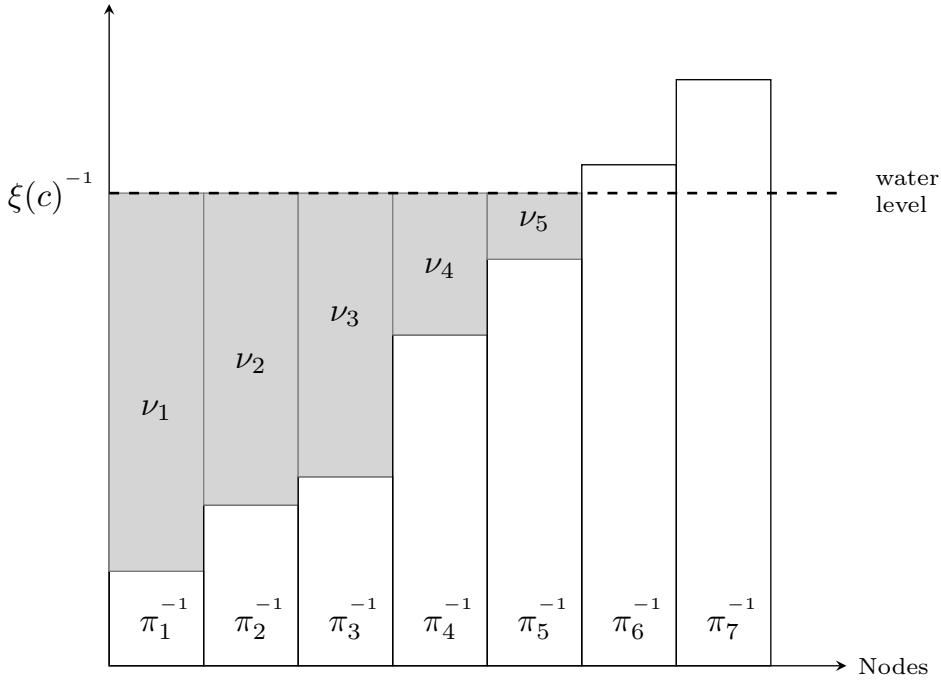
Figure 6: Water-filling example of optimal protection: $2\log(\nu_i) + 2\log(\pi_i^{-1}) = \log(\xi^{-1})$. We have assumed that $\pi_i < 1, \forall i$.

We now make some considerations on the optimal solution $\nu^*(c)$ first focusing on the role of centrality π and then on the role of parameter d.

Assume, again, that $d_i = 1$ for all i. When $\xi(c) < \pi_i^2$, the optimal protection on node i has to satisfy the following relation

$$2\log(\nu_i^*(c)) + 2\log(\pi_i^{-1}) = \log(\xi(c)^{-1}).$$

When protection is active, $\nu^*(c)$ levels out the difference created by network topology, embedded in the centrality π. In most applications, π is a centrality vector normalized to sum up to one and therefore π_1 results strictly less than 1. This fact implies that both $\log(\pi_i^{-1}), \log(\xi(c)^{-1})$ result positive quantities.

This form of the solution is called *water-filling* structure and is widely used in engineering problems. A typical example is the capacity-achieving solution for a frequency-selective channel [41, 50]. We think of it as the ground level $\log(\pi_i^{-1})$ above area i, and then flood the region with water to a depth $\log(\xi^{-1})$, as depicted in Figure 6. The total amount of water used is then $\sum_{i=1}^{m} \max\{0, \log(\xi^{-1}) - 2\log(\pi_i^{-1})\}$. We then increase the flood level until we have used a total amount

of water equal to c to obtain $\xi(c)$. The depth of water above patch i is then the optimal value $\log(\nu_i^*(c))$.

From this representation, it is clear that augmenting the amount of water through budget c will increase the amount of protection $\nu_i(c)^*$ on node i. Formally, we can conclude that $\nu_i^*(c)$ results non-decreasing in c.

Consider now a network such that π_i is equal for all the agents. Assuming a normalization on the sum of π, we then consider $\pi_i = 1/\sqrt{m}$. We now focus the attention on the role of parameter d_i.

From (5.10), we deduce that a protection is active of node i if

$$\xi(c) \leq \frac{1}{m} d_i^{-2} \, .$$

Given that $\xi(c)$ is strictly decreasing in c, higher d_i implies that an higher budget c is needed to activate a protection on node i. As a concrete examples assume that $d_j > d_i$ and that $d_k = d_i$ for all $k \neq j$. Then, it is important to notice that the optimal protection is equal on all the agents $k \neq j$.

When neither $d_i = 1$ nor $\pi_i = 1/\sqrt{m}$, for all i, the analysis becomes much more complicated because the most important agent π_1 could not be the most important with respect to the adjusted vector $\hat{\pi}$. In the next section, we will draw interesting examples showing these behaviors.

5.3 APPLICATIONS

5.3.1 *Aggregate volatility*

We recall that, in this context, the equilibrium vector is defined as

$$x = M[\nu]^{-1}\omega, \quad M = (1-\beta)(I - \beta P)^{-1} \, ,$$

where $\beta \in (0,1)$ and P is a row-stochastic matrix that represents the interaction network between firms. Throughout this section, we will assume that matrix P is the normalized adjacency matrix.

Formally, consider an economy of n firms and denote with W its adjacency matrix. We put $P = [w]^{-1}W$, where $w = W\mathbb{1}$ is the out-degree vector. If $w_i = 0$ the we set $P_{ij} = 0$ for all j. We will set $d = \mathbb{1}$ in order to focus on the role of the centrality π.

The most important performance measure in the Cobb-Douglas model is the aggregate volatility, described in our theory by $\Gamma(\nu, K)$. Assuming that the log-

productivity shocks ω are independent with mean $\mathbb{E}[\omega_i] = 0$ and variance $\mathrm{Var}(\omega_i) = \sigma_i^2 > 0$, it is possible to write aggregate volatility using the form (5.1) as

$$\Gamma(\nu, [\sigma^2]) = \sum_{i=1}^{n}(\sigma_i v_i/\nu_i)^2 . \tag{5.21}$$

We now apply the result of Theorem 5.2.2 to aggregate volatility to calculate low and high budget regimes. Assume that agents are ordered in decreasing order with respect to the centrality vector v.

If $c < \underline{c} = (n-1) + (v_1/v_2)^2$, i.e. c is in the low budget regime, then the optimal aggregate volatility results

$$f_v(\nu^*(c)) = (c - (n-1))^{-1}v_1^2 .$$

Instead, c is in the high budget regime if $c > \bar{c} = \|v\|^2/v_n^2$, and the optimal aggregate volatility results

$$f_v(\nu^*(c)) = c^{-1}\|v\|^2 .$$

It is important to notice that the optimal aggregate volatility depends on network topology through the vector v, the vector of Katz centrality. Higher is the gap between the first two most important agents, and higher is the value of $\underline{c}$. Lower is the difference among agents' importance, and lower is the value of $\bar{c}$. We also observe that optimal value $f_v(\nu^*(\bar{c}))$ is higher in networks with the highest Katz centrality norm, that is, networks that manifest the presence of central hubs and generate disparities. Regular networks are those with lowest value of $f_v(\nu^*(\bar{c}))$.

Following the analysis done in [6, 37], we now want to compare different economies studying values $\underline{c}, \bar{c}$ and $\|v\|^2$.

An important class of economies is formed by regular networks. Within the Cobb-Douglas model, a regular network implies that matrix P is doubly stochastic. A doubly-stochastic matrix P also implies that $M = (1-\beta)(I-\beta P)^{-1}$ results doubly-stochastic and Katz centrality results $v = n^{-1}\mathbb{1}$. Given that in regular networks all the agents have the same centrality $v_i = 1/n$, it results that $\underline{c} = \bar{c} = 0$.

We now consider four economies with n nodes: the cycle economy C_n, the star economy S_n, the path economy P_n, and the complete-bipartite economy $B_{n1,n2}$, such that $n_1 + n_2 = n$. All the economies are depicted in Figure 7.

We now study the high budget and the low budget regime for these four economies.

- (C_n) It is easy to see that C_n results in a regular network and that matrix P is doubly stochastic, . As highlighted before, all the agents have the same centrality and $\underline{c} = \bar{c} = 0$.
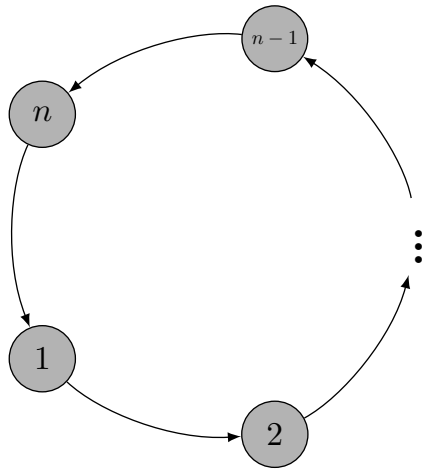
(a) Cycle economy with n agents.
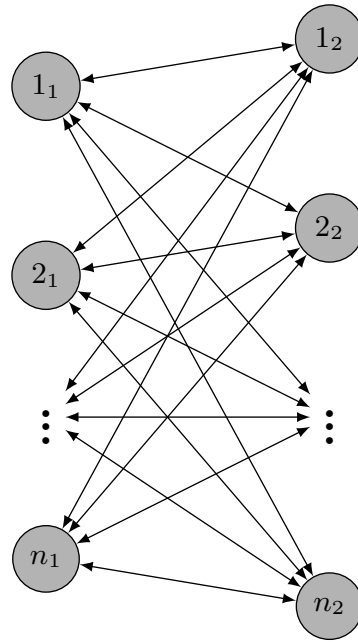
(b) Complete bipartite economy with $n1 + n2$ agents.
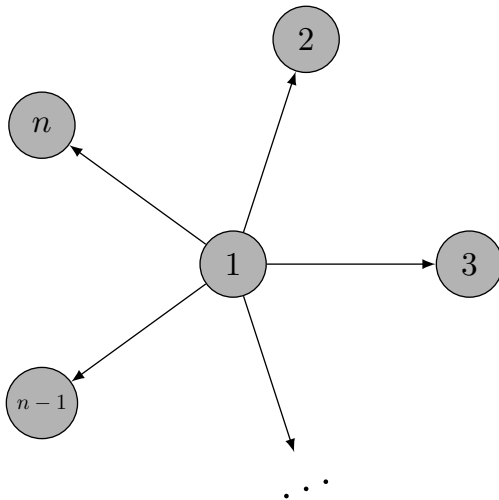
(c) Star economy with n agents.

(d) Path economy with n agents.

Figure 7: Economic networks with n nodes.

- (S_n) Denoting the hub with subscript 1 and leafs with subscript j, the centrality vector v results

$$v_1 = (1 + (n-1)\beta)/n, \quad v_j = (1-\beta)/n, \quad \|v\|^2 = \beta^2 + (1-\beta^2)/n.$$

For the star network, we have to consider only two different regimes. In the low budget regime, only node 1 is protected and persists until c reaches:

$$\underline{c} = \bar{c} = n\frac{1 + \beta^2(n-1)}{(1-\beta)^2}.$$

- (P_n) For this economy the centrality vector results

$$v_n = (1-\beta)/n, \quad v_k = v_n \sum_{j=0}^{n-k} \beta^j \text{ if } k = 2,\ldots,n-1, \quad v_1 = \sum_{j=0}^{n-1} \beta^j/n.$$

Given that all the agents have different centrality, there exist $n-1$ different regimes. The norm results

$$\|v\|^2 = v_n^2 \left[1 + \sum_{k=1}^{n-1} \left(\sum_{j=0}^{n-k} \beta^j \right)^2 \right],$$

and, therefore, the low budget regime and high budget regimes are

$$\underline{c} = (n-1) + \left(\frac{\beta}{1-\beta} + \frac{1}{1-\beta^{n-1}} \right)^2, \quad \bar{c} = 1 + \sum_{k=1}^{n-1} \frac{(1-\beta^{n-k+1})^2}{(1-\beta)^2}.$$

- $(B_{n1,n2})$ Assume that $1 \leq n_1 \leq n_2$ and denote with B_1 and B_2 the two subset of nodes. For this economy the centrality vector is

$$v_i = \frac{(1 + \beta n_2/n_1)}{n(1+\beta)} \text{ if } i \in B_1, \quad v_j = \frac{(1 + \beta n_1/n_2)}{n(1+\beta)} \text{ if } j \in B_2,$$

and the euclidean norm results

$$\|v\|^2 = \frac{n(1+2\beta) + \beta^2 \left(n_2^2/n_1^2 + n_1^2/n_2^2 \right)}{n^2(1+\beta)^2}.$$

Observe that $v_i \geq v_j$, and also in this case, there are only two regimes. In the first, the low budget regime, the optimal protection intervenes on the set B_1 allocating, equal resources all over agents in B_1. In the second, the high

budget regime, when there are enough resources, also protections on agents in B_2 start. This effect arises due to the "second order" effect of the protection on a network: protecting agents in B_1 affects agents in B_2 indirectly. The low budget regime and high budget regime result

$$\underline{c} = \overline{c} = (n-1) + \left(\frac{1 + \beta n_2/n_1}{1 + \beta n_1/n_2}\right)^2 .$$

Table 3 resumes results when $n \to \infty$ and β remains fixed. We have denoted $\gamma = \beta/(1-\beta)$ and for the bipartite economy we have assumed that $n_2 \to \infty$ and n_1 remains fixed and finite.

	$\underline{c}$	$\overline{c}$
S_n	$(n\gamma)^2$	$(n\gamma)^2$
P_n	n	$n/(1-\beta)^2$
$B_{n1,n2}$	$(n_2\beta/n_1)^2$	$(n_2\beta/n_1)^2$

Table 3: Resuming table of low budget and high budget regimes when $n \to \infty$ for the star S_n, the path P_n, and the complete bipartite $B_{n1,n2}$ economies.

5.3.2 *Stochastic inputs in opinion dynamics*

We recall that, in this context, the equilibrium vector is defined as

$$x = M[\nu]^{-1}\omega, \quad M = (I - [\lambda]P)^{-1}(I - [\lambda]),$$

where $\lambda \in \mathbb{R}^n$, $\lambda_i \in [0,1]$, for all $i = 1, \dots, m$, and P is a row-stochastic matrix that represents the social network of agents. Throughout this section, we will assume that matrix P equals the normalized adjacency matrix. Moreover, we assume $\lambda = \beta \mathbb{1}, \beta \in [0,1]$. Notice that, with this last assumption, the Cobb-Douglas model coincides with the Friedkin-Johnsen model.

In this section, we will show the differences between centralities v, ℓ, and p, and implications on the structure of the optimal protection $\nu^*(c)$. In particular, we will show a numerical example of an undirected network of $n = 10$ agents.

We start comparing centralities assuming that they are all normalized to sum up to one.

Consider the undirected network of $n = 10$ agents depicted in Figure 8. Table 4 resumes centrality v, ℓ, and p assuming $\beta = 0.5$. All centrality values have to be multiplied by 10^{-2}.
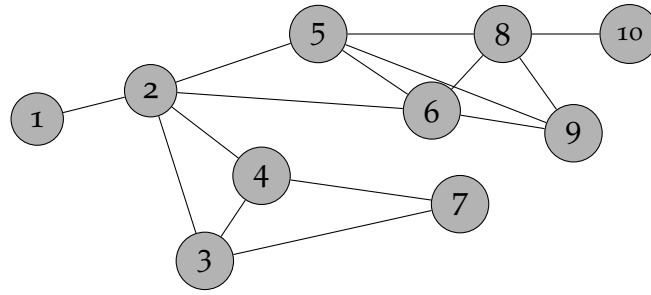
Figure 8: Undirected network with $n = 10$ nodes.

i	1	2	3	4	5	6	7	8	9	10
v	6.4	**14.3**	10.2	10.2	10.9	10.9	8.4	**12.6**	9	6.6
ℓ	8.8	**11.3**	10.2	10.2	9.9	9.9	9.7	**11.3**	9	9
p	9.4	10	**10.5**	**10.5**	9	9	10.4	**11.7**	9.5	9.7

Table 4: Centrality v, ℓ, and p of network depicted in Figure 8, assuming $\beta = 0.5$. The first column represents the agents. All centrality values have to be multiplied by 10^{-2}.

First of all, notice that agent 2 is the most important agent for v, followed by agents 8, 5, and 6. For ℓ there are two agents that are equally important, agent 2 and agent 8, followed by agents 3 and 4. For centrality p, the agent with highest centrality is agent 8, followed by agents 3 and 4. However, the fact that $v_2 > \ell_2$ implies that agent 2 has much more importance if we consider centrality v. In fact, this last centrality measures the importance of an agent as a function of the importance of its neighbors and is a more local measure because neighbors of the neighbors account for less and less. This is not the case for cycle centrality ℓ and concentration centrality p, given that both take into account a larger sphere of influence of a node.

This phenomenon is clear if we consider other agents of the network. Even if the ranking of the first two agents (agents 2 and 8) is equal for both v and ℓ, it changes starting from the third position of the ranking. In fact, it seems that centrality v gives much more importance to agents 5 and 6 because it is directly connected to the most important agents, i.e. agents 2 and 8. On the contrary, centrality ℓ gives more importance to agents 3 and 4, which are only connected to agent 2. It seems that the importance of agents 5 and 6 is incorporated in one of agents 2 and 8.

Consider now concentration centrality p. Except for node 8, this centrality produces a significantly different ranking. We recall that the concentration centrality of agent i captures how its influences are distributed across the rest of the network. A higher value of p_i means that the agent does not distribute its influence equally.

In general, it is important to notice that values in v are much more dislocated with respect to the value central value $1/n$, the empirical mean value.

We now show differences in the optimal protections applied to the network of Figure 8. Consider the Friefkin-Johnsen model (3.11) with the standing assumption of independent perturbation ω with mean $\mathbb{E}[\omega] = 0\mathbb{1}$ and diagonal covariance matrix $K = [\sigma^2], \sigma^2 = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$. The form of the optimal solution, given by Theorem 5.2.2, relates each solution to the specific centrality π, that depends on the performance considered. We analyze the difference in the optimal protection for the three objective functions $f_v, f_\ell,$ and f_p. We will denote with $\nu^v(c)$ the optimal solution of $\min_{\nu \in \mathcal{Q}_c} f_v(\nu)$, with $\nu^\ell(c)$ the optimal solution of $\min_{\nu \in \mathcal{Q}_c} f_\ell(\nu)$, and with $\nu^p(c)$ the optimal solution of $\min_{\nu \in \mathcal{Q}_c} f_p(\nu)$.

Figure 9 shows the square of three optimal performances $\nu^v(c), \nu^\ell(c),$ and $\nu^p(c)$ when $\beta = 0.5$ and for $c \in [10, 20]$. There are two main facts to highlight.

- Differences in the dispersion of centrality values are reflected in values of c for which the protections became actives on nodes, depicted with dashed vertical lines. In particular, the main difference in optimal protections is the beginning of the high budget regime, that is, the value of c such $\nu^*(c) > \mathbb{1}$. In the interval $c \in [10, 20]$, the optimal protection$\nu^v(c)$ does not even start the high budget regime. Instead, both $\nu^\ell(c)$ and $\nu^p(c)$ start the high budget regime quite fast, i.e., for $c < 13$. Protection $\nu^p(c)$ is the one that concentrates the activation in the tight interval.

- Clearly, having fewer protections active implies that protections' strength on nodes is higher. In particular, notice that the strength of $\nu^v(c)$ is much higher than the others, captured by values in the y-axis. Then, in order to minimize f_v, it is important to target the most important agents with high protection. Instead, both $\nu^\ell(c)$ and $\nu^p(c)$ distribute the available budget in a much wider way.

We conclude by stating the most important issue that arises by comparing centrality measures $v, \ell,$ and p on the same network. Optimal protections $\nu^\ell(c)$ and $\nu^p(c)$ take into account more the network as a whole than $\nu^v(c)$.
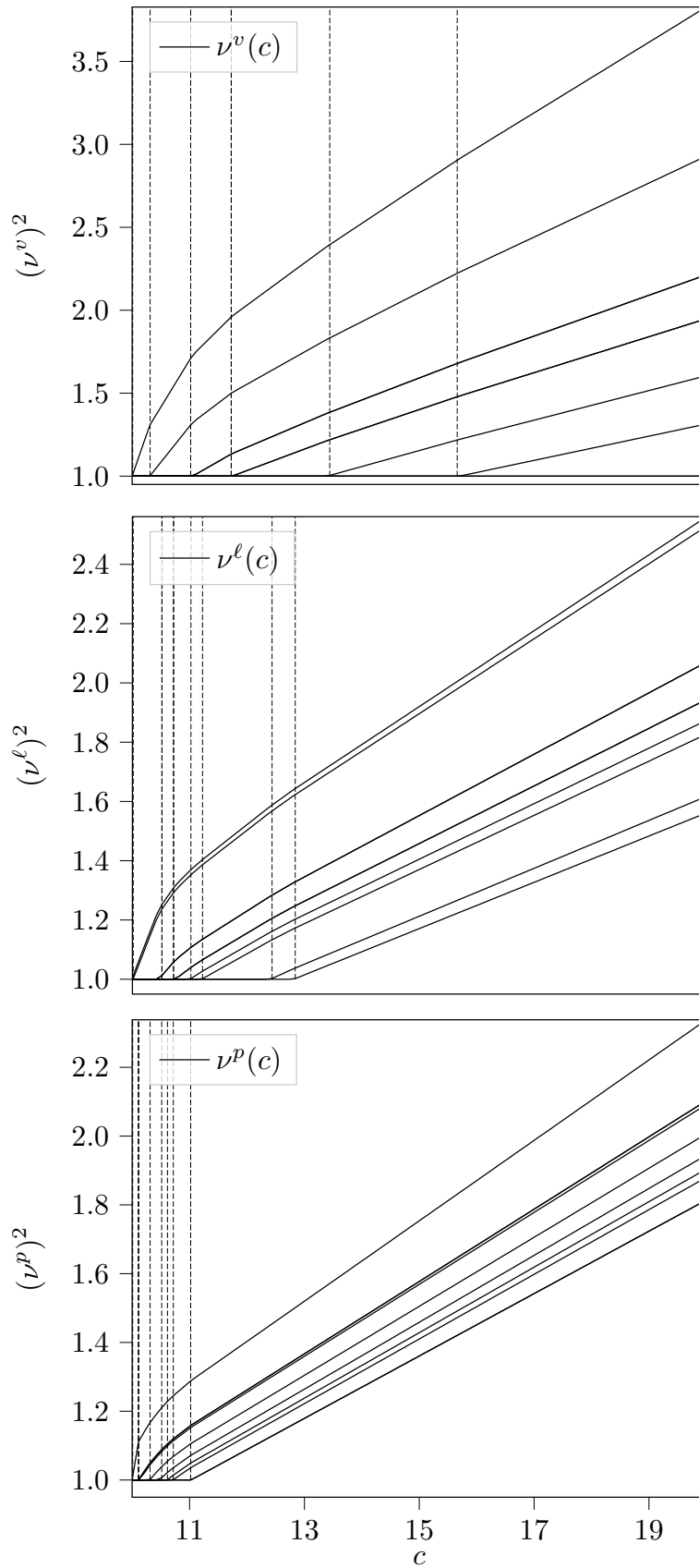
Figure 9: Optimal protections ν^v, ν^ℓ and ν^p in the case of an undirected social network of $n = 10$ nodes.

CORRELATION AMONG DISTURBANCES

In this chapter, we undertake a fundamental study of the adversarial min-max problem (4.4) under the assumption that matrix K belongs to the set Ω.

In Section 6.1, we introduce the min-max problem stating important general properties. Differently from Chapter 5, in this chapter, we analyze separately three different problems that derive from applications and depend on the nature of matrix Y. In particular, we will consider $Y = n^{-1}J$ (related to aggregate volatility), $Y = I$ (related to social welfare and absolute displacement), and $Y = (I - J)$ (related to polarization).

In Section 6.2, we study and solve the adversarial problem in the case $Y = n^{-1}J$. It is immediate to notice that, in this case, matrix $M'YM$ results in a rank-one matrix. We will show that this simplification allows studying the external minimization problem using similar tools used in Section 5.2. Similarly to Section 5.3.1, the centrality measure that naturally emerges from this analysis is the Bonacich centrality, and the optimal intervention of the defender should be derived in an explicit form.

Then, in Section 6.3, we study and solve the adversarial problem in the case $Y = I$. The minimization problem considered in this section is much more difficult with respect to the case $Y = n^{-1}J$ and could be defined as an eigenvalue optimization problem [33, 53]. Even if this class of problem is, in general, convex and hence solvable with standard optimization techniques, it is not straightforward to obtain a closed form solution. The main contribution of this section is an explicit recursive solution of the min-max problem that shows how the optimal solution for the defender is to invest its mitigation resources on all nodes if her available budget is sufficiently high or on just a subset of nodes otherwise. Emerging from the analysis is a novel network centrality measure on the set of inputs that indicates which are the most influential inputs on which the defender should mostly intervene. Part of the work described in this section has been previously published in [27].

We end this chapter with Section 6.4 by presenting and studying the min-max problem in the case $Y = I - J$. Similarly to the case $Y = I$, also in this case the minimization problem should be defined in the class of constrained spectral radius optimization. However, differently from the previous analysis, matrix $M'YM$ has negative entries and therefore complicates the study. This section shows a part

of ongoing research and examines the particular cases of a doubly-stochastic matrix M.

Throughout each section, we draw examples coming from applications, in line with the ones presented in the previous chapter, to show relevance of our solutions.

6.1 THE MODEL

In this chapter, we want to make an exhaustive study of the adversarial min-max optimization problem

$$\min_{\nu \in \mathcal{Q}_c} \max_{K \in \Omega} F_Y(\nu, K), \tag{6.1}$$

where, we recall,

$$F_Y(\nu, K) = \text{Tr}\left([\nu]^{-1}K[\nu]^{-1}M'YM\right),$$

and Ω represents the space of nonnegative positive semi-definite matrices with trace bounded by one. In particular, defining

$$K(\nu) = \underset{K \in \Omega}{\text{argmax}} \ F_Y(\nu, K),$$

Lemma 4.3.1 gives the explicit formula of $F(\nu, K(\nu))$ characterizing it as the spectral radius of matrix $[\nu]^{-1}M'YM[\nu]^{-1}$, i.e.

$$F(\nu, K(\nu)) = \rho\left([\nu]^{-1}M'YM[\nu]^{-1}\right). \tag{6.2}$$

Elements of matrix $[\nu]^{-1}M'YM[\nu]^{-1}$ depend analytically on ν and, therefore, its eigenvalues result continuous functions of ν. However, the main problem of (6.2) is that eigenvalues of $[\nu]^{-1}M'YM[\nu]^{-1}$, and hence its spectral radius, may not be differentiable as they coalesce. In this case, standard optimization techniques, such as the method of Lagrange multiplier, could not be applied [53]. In the next sections, we show that for the two measures Γ and Φ we are able to bypass this problem.

Remark. Depending on properties of matrix Y, the nature of $\rho\left([\nu]^{-1}M'YM[\nu]^{-1}\right)$ radically changes. Consider for examples $Y = I$ and $Y = J$. In the first case, the matrix $M'YM$ simply results $M'M$, and $F(\nu, K(\nu))$ represents the spectral radius of a full rank matrix. In the second case, the matrix $M'YM$ results the rank-one matrix $M'\mathbb{1}\mathbb{1}'M$ and therefore $F(\nu, K(\nu))$ could be easily represented as $\text{Tr}\left([\nu]^{-1}M'\mathbb{1}\mathbb{1}'M[\nu]^{-1}\right)$, a much more easier case than $Y = I$.

Given that the difficulty of $Y = n^{-1}J$ is much less than $Y = I$ and $Y = I - J$, in the following sections, we study (6.1) separately. We start with $Y = n^{-1}J$, where we are able to explicitly find the optimal intervention using similar tools of Chapter 5. Then, we continue with $Y = I$, a much more complex case for which we give an exact iterative solution. In the last section, we study $Y = I - J$, where we find an explicit solution to the particular case of doubly-stochastic matrix M.

6.2 AVERAGE MEASURE

In this section we solve the optimization problem

$$\min_{\nu \in \mathcal{Q}_c} \max_{K \in \Omega} n^{-1} \operatorname{Tr}\left([\nu]^{-1}K[\nu]^{-1}M'JM\right), \tag{6.3}$$

that is, problem (6.1) restricted to the case of $Y = n^{-1}J$.

We recall that $F_{n^{-1}J}$ coincides with Γ and, in this section, we reuse this last symbol to denote the objective function of (6.3). We also recall that Γ describes the aggregate volatility of an economic network. $K \in \Omega$ allows considering correlations among log-productivity shocks and not only independent shocks.

As highlighted before, in the special case $Y = n^{-1}J$ the spectral radius of $n^{-1}[\nu]^{-1}M'JM[\nu]^{-1}$ has an easy representation. Notice, in fact, that $M'JM$ is a rank-one matrix and, therefore, its spectral radius coincides with its trace.

Define the function $\gamma : \mathcal{Q}_c \to \mathbb{R}$ as

$$\gamma(\nu) = \max_{K \in \Omega} \Gamma(\nu, K), \quad \nu \in \mathcal{Q}_c,$$

and let v be the vector defined as

$$v = n^{-1}M'\mathbb{1},$$

in line with definition of Katz centrality (5.5). The next result shows the nature of the function γ.

Corollary 1. *For every ν in $\mathcal{Q}_c$, it holds*

$$\gamma(\nu) = \sum_{i \in \mathcal{S}} (v_i / \nu_i)^2. \tag{6.4}$$

Proof. The proof follows by the fact that spectral radius of a rank-one matrix is equal to its trace. $\square$

Function γ results in a weighted sum of separable functions. Moreover, (6.4) coincides with the objective function of problem (5.6) when matrix K is equal to the identity matrix.

We now present the solution to the outer minimization problem of (6.3).

If $K \in K(\nu)$, the starting optimization problem (6.3) becomes

$$\min_{\nu \in \mathcal{Q}_c} \sum_{i \in \mathcal{S}} (v_i/\nu_i)^2 . \tag{6.5}$$

Since $\mathcal{Q}_c$ is convex, then the minimum of γ on such a set is unique. We denote it by $\nu^*(c)$ to remember its dependence on the budget c.

Before stating the main result of this section we need to introduce some more concepts. Similarly to the analysis done in Section 5.2 we introduce the vector $\hat{v} = (v_1/d_1^2, v_2/d_2^2, \ldots, v_m/d_m^2)$, assuming a decreasing order in its elements, i.e $\hat{v}_1 \geq \hat{v}_2 \geq \cdots \geq \hat{v}_m$. Changing π with v, the water-level function (5.8) becomes the function $T_{v,d}(\xi) : (0, +\infty) \to \mathbb{R}$ given by

$$T_{v,d}(\xi) := \sum_{i=1}^{m} d_i^2 \max \{1, \hat{v}_i/\xi\} . \tag{6.6}$$

We notice again that $T_{v,d}$ is continuous, strictly decreasing in $(0, \hat{v}_1]$, and satisfies

$$\lim_{\xi \to 0+} T_{v,d}(\xi) = +\infty, \quad T_{v,d}(\hat{v}_1) = \|d\|^2.$$

This implies that for every $c \geq \|d\|^2$, it is well defined $\xi(c) := T_{v,d}^{-1}(c)$. Let $k(c)$ be the maximum index such that $\hat{v}_{k(c)} > \xi(c)$.

The following result holds true.

Theorem 6.2.1. *Let $d \in \mathbb{R}_{++}$ and $c \geq \|d\|^2$.*

It holds

$$\min_{\nu \in \mathcal{Q}_c} \gamma(\nu) = \xi(c) \sum_{i \leq k(c)} v_i + \sum_{i > k(c)} (v_i/d_i)^2 , \tag{6.7}$$

and the optimum value for $\nu(c)$ is reached by

$$\nu_i^*(c) = \begin{cases} \sqrt{v_i/\xi(c)} & \text{if } i \leq k(c) \\ d_i & \text{otherwise} . \end{cases} \tag{6.8}$$

Proof. During the proof we will use ν_i instead of $\nu_i(c)$ to simplify the notation. Monotonicity properties of γ imply that, necessarily, $\|\nu\|^2 = c$. Using classical Lagrangian multipliers techniques, we obtain the following equations:

$$\begin{cases} 2\mu \nu_i - 2v_i^2/\nu_i^3 \geq 0, & i = 1, 2, \ldots m \\ \|\nu\|^2 = c, \\ (\mu - v_i^2/\nu_i^4) (\nu_i - d_i) = 0, & i = 1, 2, \ldots m , \end{cases} \tag{6.9}$$

where $\mu > 0$ is the Lagrange multiplier associated to the constraint $\|\nu\|^2 \leq c$. We define $\xi = \sqrt{\mu}$ and we proceed as follows.

Then, the first bulk of equations yield

$$\nu_i \geq \sqrt{v_i/\xi}. \tag{6.10}$$

Similarly to the proof of Theorem 5.2.2 we now analyze the nature of (6.21) as a function of ξ.

The lower bound constraint $\nu_i \geq d_i$ imposes that $\xi \leq v_i/d_i^2$, for all i.

Assume first that $\xi < v_i/d_i^2$. Then, ν_i results greater than d_i. Hence, to satisfy the third equation of (6.9) it results

$$\nu_i = \sqrt{v_i/\xi_i}.$$

Assume now $\xi = v_i/d_i^2$. Then, (6.10) results $\nu_i \geq d_i$. We now prove that $\nu_i = d_i$. Assume by contradiction that $\nu_i > d_i = \sqrt{v_i/x_i}$. Then, the third equation of (6.9) is never satisfied and therefore we conclude that $\nu_i = d_i$.

Hence, we have obtained that the optimal solution could be written as

$$\nu_i = \max\left\{d_i, \sqrt{v_i/\xi}\right\}.$$

Plugging the optimal solution ν in the constraint $\sum_i \nu_i^2 = c$ we obtain

$$\sum_i \max\left\{d_i^2, v_i/\xi\right\} = c.$$

Notice that the left hand side is $T_{v,d}(\xi)$. Given that $T_{v,d}$ is invertible there exists an optimal ξ defined as $\xi = T_{v,d}^{-1}(c), \forall c > \|d\|^2$. Therefore, we could write optimal solution as

$$\nu_i = \max\left\{d_i, \sqrt{v_i/T_{v,d}^{-1}(c)}\right\}.$$

Plugging this value in (6.5) gives the result. $\qquad\square$

We now comment on the results obtained. Similarly to the results of Theorem 5.2.2, the structure of the optimums we have found shows how the optimal protections are, in general, concentrated on a proper subset of nodes. We then analyze various regimes depending on the chosen budget cost c. For simplicity of exposition, we assume that $d = \mathbb{1}$ in order to mainly focus on the role of Katz centrality vector v. The vector v is then assumed ordered with a decreasing order.

1. The low budget regime exists if c is such that $c < T_{v,1}(v_2) = (m-1) + v_1/v_2$. In this case, we get from (6.7) that the optimal value is given by

$$\gamma(\nu^*(c)) = (c - (m-1))^{-1} v_1 + \sum_{i \geq 2} v_i^2. \tag{6.11}$$

2. The high budget regime exists if c is such that $c > T_{v,1}(v_m) = \sum_{i \in \mathcal{S}} (v_i / v_m)$. In this regime, we get from (6.7) that the optimal value is given by

$$\gamma(\nu^*(c)) = c^{-1} \sum_{i \in \mathcal{S}} v_i . \qquad (6.12)$$

Same considerations drawn for results of Theorem 5.2.2 are still valid. The main difference to notice with respect to the objective function f_v is that $\gamma(\nu^*(c))$ also consider the uncontrolled part, characterized by $\sum_{i>k(c)} v_i^2$. This difference is motivated by the nature of K. In fact, when K is a diagonal matrix, the effect of the worst perturbation, described in Lemma 5.2.1, is focused on one agent only. Differently, when K is a full matrix, the effect of the worst perturbation influences also other agents of the network.

Remark. In most applications, the vector v is assumed to sum up to one. Therefore, result (6.12) states that all networks perform equally in the high budget regime.

In the next part, we continue the analysis done on the Cobb-Douglas model, started in Section 5.3.1, in order to highlight differences between $\nu^*(c)$ when K is diagonal and when is not.

6.2.1 *Application: aggregate volatility with correlated shocks*

We compute the low and high budget regimes (6.11) and (6.12) for the economies described by Figure 7. As stated in the previous remark, given that $\mathbb{1}'v = 1$, the optimal value $\gamma(\nu^*(c))$ results equal among all networks. We also observe that the optimal protection (6.8) on agent i depends on v_i, while, in the case of independent perturbations, depends on v_i^2.

- (C_n) Also in this case, the optimal protection on cycle economy, and more in general on regular economies, is equal for all the agents. This is motivated by the fact that, in a regular economy, all the agents have the same centrality v and, therefore, the optimal protection consists always in protecting all the agents, equally dividing the available budget c. Hence, low budget and high budget regimes coincide and are equal to 0.

- (S_n) In the star economy the low budget regime, that coincides with high budget regime, results
$$\underline{c}_S = \bar{c}_S = n/(1 - \beta) .$$

- (P_n) In the path economy the low budget and high budget regimes result

$$\underline{c}_P = (n-1) + \left(\frac{\beta}{1-\beta} + \frac{1}{1-\beta^{n-1}} \right), \qquad \overline{c}_P = n/(1-\beta).$$

- $(B_{n1,n2})$ In the complete bipartite economy the low budget and high budget regimes result

$$\underline{c}_B = \overline{c}_B = n \frac{(1+\beta)}{(1+\beta n_1/n_2)}.$$

We now compare these results with ones of Section 5.3.1. To differentiate budget regimes, we indicate $\underline{c}^\sigma, \overline{c}^\sigma$ low and high budget regimes of Section 5.3.1 and simply $\underline{c}, \overline{c}$ low and high budget regimes of this section.

We start comparing low budget regimes keeping n and β fixed. We observe that $\underline{c} \leq \underline{c}^\sigma$ for the three economies that are not regular, i.e. the star, the path, and the bipartite economies. This means that to protect the network from a correlated perturbation, it is better to divide the available budget among a larger set of nodes.

In line with observations on the low budget regimes, it is true that $\overline{c} \leq \overline{c}^\sigma$. When K is full, the optimal intervention starts before protecting all the agents in the network. This fact is motivated by correlations between perturbations.

Table 5 resumes values $\underline{c}$ and $\overline{c}$ when $n \to \infty$ and β is fixed. It also shows low and high budget regimes $\underline{c}^\sigma, \overline{c}^\sigma$ calculated in Section 5.3.1. For the bipartite economy, we have assumed that $n_2 \to \infty$ and n_1 remain fixed and finite.

	$\underline{c}$	$\underline{c}^\sigma$	$\overline{c}$	$\overline{c}^\sigma$
S_n	$n/(1-\beta)$	$(n\beta)^2/(1-\beta)^2$	$n/(1-\beta)$	$(n\beta)^2/(1-\beta)^2$
P_n	n	n	$n/(1-\beta)$	$n/(1-\beta)^2$
$B_{n1,n2}$	$n(1+\beta)$	$(n\beta)^2$	$n(1+\beta)$	$(n\beta)^2$

Table 5: Resuming table of low and high budget regimes when $n \to \infty$ for the star S_n, the path P_n, and the complete bipartite $B_{n1,n2}$ economies.

6.3 MAGNITUDE MEASURE

In this section we solve the optimization problem

$$\min_{\nu \in \mathcal{Q}_c} \max_{K \in \Omega} \operatorname{Tr}\left([\nu]^{-1} K [\nu]^{-1} M'M \right), \tag{6.13}$$

that is, problem (6.1) restricted to the case of $Y = I$. We recall that F_I coincides with Ψ and, in this section, we reuse this last symbol to denote the objective function of (6.13).

In this chapter, we define matrix

$$H = M'M \in \mathbb{R}_+^{m \times m} \tag{6.14}$$

that results a nonnegative symmetric matrix such that $H_{ij} > 0$ if and only if there exists at least a regular agent k from which both sources i and j are reachable in the graph $\mathcal{G}$. In fact, we may interpret H as the weighted adjacency matrix of a new undirected graph $\mathcal{H} = (\mathcal{S}, \mathcal{F})$ with node set $\mathcal{S}$, whereby i and j are linked by an undirected link of weight H_{ij} if and only if there exists at least a regular agent k that is (possibly indirectly) influenced by both sources i and j in the original network $\mathcal{G}$. We shall denote by

$$\pi = \frac{1}{\sum_{i,j} H_{ij}} H\mathbb{1}, \tag{6.15}$$

the normalized degree centrality vector of this graph.

To maintain a simplicity of exposition, we shall make the following assumption.

Assumption 2. *The matrix H is irreducible.*

Assumption 2 is equivalent to requiring that the graph $\mathcal{H}$ is connected. Notice that a sufficient condition for Assumption 2 to be satisfied is that $A + A'$ is irreducible. From the network point of view, the previous assumption implies that the graph has to be weakly connected, that is, replacing all directed edges with undirected ones generates a connected graph.

We make some more comments on Assumption 2. Consider $\mathcal{H}$ composed by $k \geq 2$ connected components $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k$. Then, matrix H results reducible and, given that it is symmetric, it must exists a block diagonal representation

$$H = \begin{pmatrix} H_{\mathcal{S}_1} & 0 & \cdots & 0 \\ 0 & H_{\mathcal{S}_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & H_{\mathcal{S}_k} \end{pmatrix}, \tag{6.16}$$

where the set of agents $\mathcal{S}$ has been partitioned in k subsets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ such that $\mathcal{H}_i = (\mathcal{S}_i, \mathcal{E}_i)$, where $\mathcal{E}_i$ is the set of edges of graph $\mathcal{H}$ that connect agents in $\mathcal{S}_i$. The case of $\mathcal{H}$ composed of multiple connected components will be commented on later.

6.3.1 *Weakly connected network*

In this section, we analyze the case of irreducible matrix H.

We first set some further notation. We approach the min-max problem (4.4) by first analyzing the inner maximization problem. To this aim, we define the function $\phi : \mathcal{Q}_c \to \mathbb{R}$ as

$$\phi(\nu) = \max_{K \in \Omega} \Phi(\nu, K), \quad \nu \in \mathcal{Q}_c.$$

The next result gathers some important facts on the function ϕ.

Lemma 6.3.1. *Let $d \in \mathbb{R}^m_{++}$ and $c \geq \|d\|^2$.*
  Then, for every ν in $\mathcal{Q}_c$, we have

(i) $\phi(\nu) = \rho\left([\nu]^{-1} H [\nu]^{-1}\right) = \rho\left(M [\nu]^{-2} M'\right).$

If Assumption 2 is satisfied, then:

(ii) $\phi(\nu)$ *is a simple eigenvalue of $[\nu]^{-1} H [\nu]^{-1}$*

(iii) $\phi(\nu)$ *is strictly convex in ν.*

(iv)
$$\frac{\partial}{\partial \nu_i} \phi(\nu) = -2(M'z)_i^2 / \nu_i^3 \,, \tag{6.17}$$

where z is the dominant eigenvector of $M [\nu]^{-2} M'$ associated to the eigenvalue $\phi(\nu)$.

Proof. Appendix 8. $\qquad\square$

Since $\mathcal{Q}_c$ is convex, Lemma 6.3.1 implies that the minimum of ϕ on such a set is unique. We denote it by $\nu^*(c)$ to remember its dependence on the budget c.

We first solve the min-max problem relatively to the unconstrained case where we drop the lower bound conditions expressed through d. We define, for every positive scalar c,

$$\mathcal{Q}_c^0 = \{\nu \in \mathbb{R}^m \mid \nu_i > 0, \ \|\nu\|^2 \leq c\}, \tag{6.18}$$

and we consider

$$\min_{\nu \in \mathcal{Q}_c^0} \max_{K \in \Omega} \Phi(\nu, K) = \min_{\nu \in \mathcal{Q}_c^0} \phi(\nu). \tag{6.19}$$

From Lemma 6.3.1 we deduce that the minimum above is unique and we indicate it as $\nu^0(c)$. An explicit form of $\nu^0(c)$ is presented below.

Proposition 4. *For every $c > 0$, we have that*

$$\nu^0(c) = c\sqrt{\pi}, \quad \min_{\nu \in \mathcal{Q}_c^0} \phi(\nu) = \phi(c\sqrt{\pi}) = \frac{\mathbb{1}'H\mathbb{1}}{c} \,.$$

Proof. Monotonicity properties of ϕ (see relations (6.17)) imply that, necessarily, $\|\nu^0\|^2(c) = c$. Using the explicit expression for the derivative of the objective function in 6.17 and classical Lagrangian multipliers techniques, we obtain the following equations:

$$\begin{cases} -2\nu_i^{-3}(M'z)_i^2 + 2\mu\,\nu_i = 0 & i \in \mathcal{S} \\ \|\nu\|^2 = c, \\ M[\nu]^{-2}M'z = \rho z \end{cases} \tag{6.20}$$

where μ is the Lagrangian multiplier, $\rho = \phi(\nu)$ is the value function and z is the positive dominant eigenvector of $M[\nu]^{-2}M'$. We proceed as follows. The first bulk of equations yield

$$\mu^{1/2}[\nu]^2\mathbb{1} = M'z. \tag{6.21}$$

Substituting in the third equation, we obtain

$$\mu^{1/2}M\mathbb{1} = \rho z. \tag{6.22}$$

As $\rho > 0$ and $z > 0$, we derive from (6.21) and (6.22) that

$$\nu = \mu^{-1/2}M'z = \rho^{-1}M'M\mathbb{1}. \tag{6.23}$$

The fact that $\|\nu\|^2 = c$ yields the thesis. $\qquad\square$

We now notice that the unconstrained solution $\nu^0(c)$ computed in Proposition 4 satisfies

$$\nu^0(c) \in \mathcal{Q}_c \iff c \geq c^0 = \max_{i=1}^{m} \frac{d_i^2}{\pi_i}.$$

For such values of c, in consideration of the fact that $\mathcal{Q}_c \subseteq \mathcal{Q}_c^0$, we have that the solution found also solves the constrained minimum problem. We gather this in the following result.

Theorem 6.3.2. *Let $H = M'M, M \in \mathbb{R}_+^{n\times m}$ satisfies Assumption 2.*
Then, for every $d \in \mathbb{R}_{++}^m$ and $c \geq c^0$ we have

$$\nu^*(c) = c\sqrt{\pi}, \quad \phi(\nu^*(c)) = \frac{\mathbb{1}'H\mathbb{1}}{c}. \tag{6.24}$$

In the rest of the section, we refer to the range $c \geq c^0$ as to the *high budget* regime. We recall one more time that the threshold value c^0 depends on the topology of the network, through π, but also on the lower bound vector d.

Remark. Assume that M is row-stochastic and consider $Y = I$. Then, the new centrality π defined in (6.15) coincides with v, defined in (5.5). Optimal solutions of the min-max problem (4.4) results in two different centrality dependent structure:

- if $K \in \mathbb{D}^+$, then $\nu^*(c)$ depends on concentration centrality ℓ;

- if $K \in \Omega$ and $c \geq c^0$, then $\nu^*(c)$ depends on Katz centrality v.

When c instead satisfies $\|d\|^2 \leq c < c^0$, we have that $\nu^0(c) \notin \mathcal{Q}_c$ and, consequently, the solution of the minimum constrained problem exhibits one or more components saturated to their lower bound level, namely $\nu_i^*(c) = d_i$ for some $i \in \mathcal{S}$. For later use, we define the set

$$\mathcal{U}_c = \{i \in \mathcal{S} \mid \nu_i^*(c) > d_i\}$$

that we call the set of active protections at budget level c.

As a pivot to study this more general case, we investigate a minimum problem where we assume that some of the variables ν_i are constrained to their lower bound level d_i, while the others are totally unconstrained.

We first set some notation. We assume agents to be split into two disjoint subsets $\mathcal{U}, \mathcal{W}$:

$$\mathcal{S} = \mathcal{U} \cup \mathcal{W}.$$

Agents in $\mathcal{U}$ are unconstrained while agents in $\mathcal{W}$ are the constrained ones. We define the new set of variables

$$\mathcal{Q}_c^{\mathcal{U}} = \{\nu \in \mathbb{R}_+^m \mid \nu_i = d_i \; \forall i \in \mathcal{W}, \; \|\nu\|^2 \leq c\},$$

and we split, accordingly,

$$M = \left(\begin{array}{cc} M_{\mathcal{U}} & M_{\mathcal{W}} \end{array} \right).$$

We want to study

$$\min_{\nu \in \mathcal{Q}_c^{\mathcal{U}}} \phi(\nu). \tag{6.25}$$

As $\mathcal{Q}_c^{\mathcal{U}}$ is still convex, Lemma 6.3.1 yields that the minimum above is unique. We denote it as $\nu^{\mathcal{U}}(c)$ and we explicitly derive it using first order conditions as in the proof of Proposition 4.

Proposition 5. *Let $H = M'M, M \in \mathbb{R}_+^{n \times m}$ satisfies Assumption 2. We consider a partition $\mathcal{S} = \mathcal{U} \cup \mathcal{W}$ where $\mathcal{U} \neq \emptyset$ and we fix $d \in \mathbb{R}_{++}^{\mathcal{W}}$. Then, for every $c \geq \|d\|^2$, the solution $\nu^{\mathcal{U}}(c)$ is completely described by these relations:*

$$\begin{cases} \nu^{\mathcal{U}}(c) = \left(M_{\mathcal{U}}' \left(\rho I - M_{\mathcal{W}}[d]^{-2} M_{\mathcal{W}}' \right)^{-1} M_{\mathcal{U}} \mathbb{1}, d \right) \\ \phi(\nu^{\mathcal{U}}(c)) = \rho \\ \mathbb{1}' M_{\mathcal{U}}' \left(\rho I - M_{\mathcal{W}}[d]^{-2} M_{\mathcal{W}}' \right)^{-1} M_{\mathcal{U}} \mathbb{1} = c - \|d\|^2. \end{cases} \tag{6.26}$$

Proof. Appendix 8. □

The following is the key technical result of this paper.

Theorem 6.3.3. *Let $H = M'M$, $M \in \mathbb{R}_+^{n \times m}$ satisfies Assumption 2. Fix $d \in \mathbb{R}_{++}^m$. Then, the function $\nu^* : \mathcal{Q}_c \to \mathbb{R}_{++}^m$ is continuous and (entrywise) non decreasing.*

Proof. Appendix 8. □

Using the above result, we can give a complete description of the optimal solution $\nu^*(c)$. This is the content of the following result that follows directly from Theorem 6.3.3.

Corollary 2. *Let $H = M'M$, $M \in \mathbb{R}_+^{n \times m}$ satisfies Assumption 2. Fix $d \in \mathbb{R}_{++}^m$. Then, there exists a finite sequence of points*

$$c^0 = \max_{i=1}^{m} \frac{d_i^2}{\pi_i} > c^1 > \cdots > c^s = \mathbb{1}'d$$

and subsets $\mathcal{S} \supsetneq \mathcal{U}^0 \supsetneq \mathcal{U}^1 \supsetneq \cdots \supsetneq \mathcal{U}^{s-1}$ such that

$$\mathcal{U}_c = \begin{cases} \mathcal{S} & \text{if } c > c^0 \\ \mathcal{U}^k & \text{if } c^{k+1} < c \le c^k, \ k = 0, \ldots, s-1 \end{cases}$$

$$\nu^*(c) = \begin{cases} \nu^0(c) & \text{if } c > c^0 \\ \nu^{\mathcal{U}^k}(c) & \text{if } c^{k+1} < c \le c^k, \ k = 0, \ldots, s-1. \end{cases}$$

This result says that the optimal protection vector ν^* exhibits a 'water-filling' structure as a function of the budget c. As we had already noticed, above the threshold c^0, ν^* coincides with the minimum of the unconstrained case. At $c = c^0$ one or more components saturate at their lower bound d_i and for an interval of values $]c^1, c^0]$ all remaining components remain strictly above the corresponding lower bound and the optimum ν^* coincides with the solution of problem (6.25) with $\mathcal{U} = \mathcal{U}_{c^0}$. At c^1 more components will saturate and will remain stable in an interval $]c^2, c^1]$ and so on.

We notice that the specific form of the solutions of problem (6.25) can be used to make Corollary 2 an effective recursive characterization of the optimal solution. Notice, in particular, that the sequence of thresholds c^k and the subsets $\mathcal{U}^k$ can be recursively computed through the following characterization

$$\begin{aligned} c^{k+1} &= \inf\{c \le c^k \mid \nu_i^{\mathcal{U}^k}(c) > d_i \ \forall i \in \mathcal{U}^k\} \\ \mathcal{U}^{k+1} &= \{i \in \mathcal{S} \mid \nu_i^{\mathcal{U}^k}(c^{k+1}) > d_i\}. \end{aligned} \qquad (6.27)$$

1: **procedure** ITERATIVE SOLUTION(M, d)
2:     Set $k = 0$,
3:     $c^k = \max_{1 \leq i \leq m} d_i^2/\pi_i$, and $\mathcal{U}^k = \{i : c^k \pi_i > d_i\}$
4:     **while** $|\mathcal{U}^k| > 0$ **do**
5:         Calculate $\phi(\nu^{\mathcal{U}^k})$ and $\nu^{\mathcal{U}^k}(c)$ through (6.26)
6:         $k = k + 1$
7:         Update c^k and $\mathcal{U}^k$ using (6.27)
8:         $\nu^*(c) = \nu^{\mathcal{U}^{k-1}}(c)$ for $c \in [c^k, c^{k-1}]$
9:     **end while**
10: **end procedure**

Figure 10: Iterative procedure to calculate $\nu^*(c)$.

Remark. Assume $d = \mathbb{1}$. If matrix M is doubly-stochastic, then the optimal protection results $\nu^*(c) = c/\sqrt{m}\mathbb{1}$. This finding is in line with the results of Section 6.2 and stresses that when agents are equally important with respect to the centrality π, the optimal protection will be equal among all the agents.

Assume now that matrix M is doubly-stochastic and that $d \neq \mathbb{1}$. Then, the iterative solution of Corollary 2 is much more complicated than (5.9). In particular, values $c^k, k \geq 1$ are hidden and could, in general, be different from d_i^2/π_i. The next example wants to gain some insight into this specific issue.

Example. Consider an undirected cycle graph C_n with n nodes and described by the adjacency matrix $W \in \mathbb{R}^{n \times n}$. Define $M = (1 - \lambda)(I - \lambda P)^{-1}$, where P is the normalized weight matrix and $\lambda \in (0, 1)$. Given that C_n is regular, M results doubly-stochastic and $\pi_i = 1/n$.

Assume $d_j = 2$ and $d_i = 1$ for all $i \neq j$. Figure 11 shows the behavior of the optimal protection $\nu^*(c)$ constructed with the iterative solution shown in Figure 10 for an undirected cycle graph of $n = 11$ nodes and $\lambda = 0.5$. Parameter d is such that $d_6 = 2$ and $d_i = 1$, $i \neq 6$. Values of $\nu^*(c)$ when $c = 16$ are reported in the following table:

i	$1, 11$	$2, 10$	$3, 9$	$4, 8$	$5, 7$	6
$\nu^*(c)$	1.529	1.526	1.517	1.493	1.433	2

It is important to notice that protection on nodes 1 and 11, the two nodes most far from node 6, is higher than protection on other nodes. More in general, protection is lower on nodes nearer to node 6. This phenomenon shows that the difference in parameters highly influences the nature of $\nu^*(c)$.
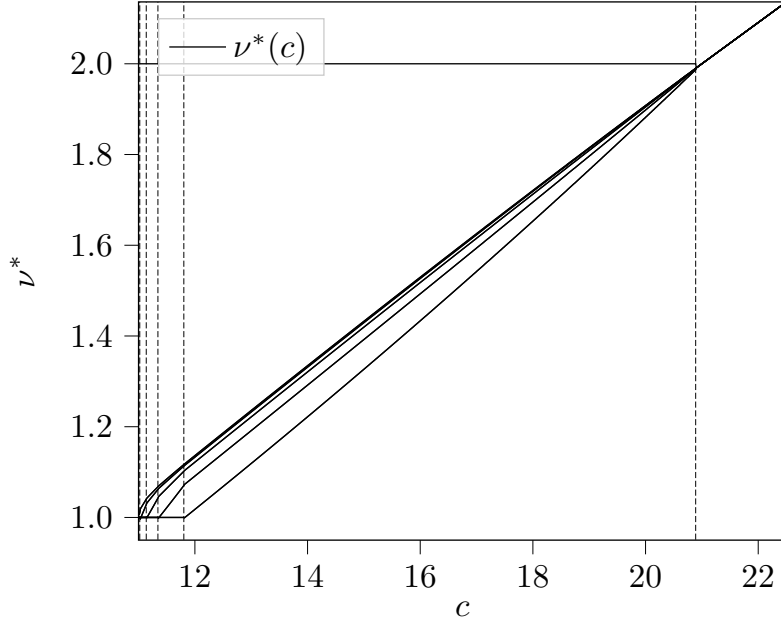
Figure 11: Optimal protection $\nu^*(c)$ on an undirected cycle graph of $n = 11$ nodes with $\lambda = 0.5$, $d_6 = 2$, and $d_i = 1$ for all $i \neq 6$.

6.3.2 *Multiple connected components*

We end this section by commenting on the generalization to the case of multiple components of graph $\mathcal{H}$. Specifically, we explicitly solve (6.13) in the unconstrained case $\nu \in \mathcal{Q}_c^0$ when $\mathcal{H}$ represents a graph composed of two weakly connected components. We show that the solution to problem (6.19) is exactly equal to the one presented in Proposition 4 and that we can generalize previous results.

As previously pointed out, when the graph $\mathcal{H}$ is composed of $k \geq 2$ components, matrix H results reducible and writable in the diagonal block form (6.16). Given that Assumption 2 is not valid, the spectral radius of $M[\nu]^{-2}M'$ could not results simple for some $\nu \in \mathcal{Q}_c$, and, therefore, we could not use item (iv) of Lemma 6.3.1 to calculate the derivative of $\rho\left(M[\nu]^{-2}M'\right)$. However, given the particular block diagonal structure of H, we show how to bypass this problem.

In order to keep the presentation as simple as possible, we assume that H is a block diagonal matrix composed by two blocks

$$H := \begin{pmatrix} H_{\mathcal{S}_1} & 0 \\ 0 & H_{\mathcal{S}_2} \end{pmatrix}, \tag{6.28}$$

and that each subset $\mathcal{S}_i$ has cardinality m_i such that $m_1 + m_2 = m$. The generalization to the case of H composed by $k \geq 2$ diagonal blocks is straightforward.

Notice that also matrix $M[\nu]^{-2}M'$ admits the block diagonal structure

$$M[\nu]^{-2}M' = \begin{pmatrix} M_{\mathcal{S}_1}[\nu_{\mathcal{S}_1}]^{-2}M'_{\mathcal{S}_1} & 0 \\ 0 & M_{\mathcal{S}_2}[\nu_{\mathcal{S}_2}]^{-2}M'_{\mathcal{S}_2} \end{pmatrix},$$

where $\nu_{\mathcal{S}_i} \in \mathbb{R}^{m_i}$ is the sub-vector obtained from ν by selecting elements ν_j, such that $j \in \mathcal{S}_i$.

We now set some further notation. We will denote by $\rho_j(\nu_{\mathcal{S}_j})$ the largest eigenvalue of the block matrix $M_{\mathcal{S}_i}[\nu_{\mathcal{S}_i}]^{-2}M'_{\mathcal{S}_i}$, noticing that $\rho_1(\nu_{\mathcal{S}_1})$ should be, in general, different from $\rho_2(\nu_{\mathcal{S}_2})$. Moreover, we highlight that the largest eigenvalue of matrix $M[\nu]^{-2}M'$ could be defined as

$$\phi(\nu) = \max\{\rho_1(\nu_{\mathcal{S}_1}), \rho_2(\nu_{\mathcal{S}_2})\}.$$

Remark. It is important to notice that items (i) and (iii) of Lemma 6.3.1 are still valid. We now comment items (ii) and (iv) of the same lemma.

(ii) Since $M[\nu]^{-2}M'$ is a nonnegative matrix, the dominant eigenvalue of each block is simple [10, Theorem 1.4]. Therefore, the algebraic multiplicity of $\phi(\nu)$ coincides with the number of times $\phi(\nu)$ is the dominant eigenvalue of a block. Given that $M[\nu]^{-2}M'$ is symmetric, then the geometric multiplicity coincides with the algebraic multiplicity. This implies that $\phi(\nu)$ is semi-simple.

(iv) Even if we could not use item (iv) of Lemma 6.3.1 to calculate the derivative of $\phi(\nu)$, we could calculate the derivative of $\rho_j(\nu_{\mathcal{S}_j})$ for each connected component j. Formally,

$$\frac{\partial}{\partial \nu_i}\rho_j(\nu_{\mathcal{S}_j}) = \begin{cases} -2(M_{\mathcal{S}_j}z_{\mathcal{S}_j})^2/\nu_i^3, & \text{if } i \in \mathcal{S}_j \\ 0, & \text{otherwise} \end{cases}$$

where $z_{\mathcal{S}_j} \in \mathbb{R}^{m_j}$ is the positive eigenvector associated to $\rho_j(\nu_{\mathcal{S}_j})$.

Since $\mathcal{Q}_c$ is convex, Lemma 6.3.1 implies that the minimum of ϕ on such a set is unique. We denote it by $\nu^*(c)$ to remember its dependence on the budget c.

We are now ready to solve the min-max problem related to the unconstrained case (6.19). Again, from Lemma 6.3.1 we deduce that the minimum above is unique, and we indicate it as $\nu^0(c)$. An explicit form of $\nu^0(c)$ is presented below.

Proposition 6. *Let $H = M'M, M \in \mathbb{R}^{n \times m}$ satisfies the block diagonal structure (6.28). For every $c > 0$, we have that*

$$\nu^0(c) = c\sqrt{\pi}, \quad \min_{\nu \in \mathcal{Q}_c^0} \phi(\nu) = \rho_1(c\sqrt{\pi_{\mathcal{S}_1}}) = \rho_2(c\sqrt{\pi_{\mathcal{S}_2}}) = \frac{\mathbb{1}'H\mathbb{1}}{c}.$$

Proof. We introduce the slack variable ξ and we reformulate problem (6.19) as follows

$$\min_{\xi \in \mathbb{R}, \nu \in \mathcal{Q}_c^0} \xi \tag{6.29}$$
$$\text{s.t.} \quad \rho_j(\nu_{\mathcal{S}_j}) \leq \xi, \quad j = 1, 2.$$

Therefore, we could use standard Lagrangian method to find the optimal solution $\nu^*(c)$.

Monotonicity properties of ϕ (see relations (6.17)) imply that, necessarily, $\|\nu^0\|^2(c) = c$. Using the explicit expression for the derivative of the objective function in (6.17) and classical Lagrangian multipliers techniques, we obtain the following equations:

$$\begin{cases} -2\beta_j \nu_i^{-3}(M'_{\mathcal{S}_j} z_{\mathcal{S}_j})_i^2 + 2\mu\,\nu_i = 0, & i \in \mathcal{S}_j, \ j = 1, 2 \\ \|\nu\|^2 = c, \\ \beta_j\,(\phi_j(\nu) - \xi) = 0, & j = 1, 2 \\ M_{\mathcal{S}_j}[\nu]^{-2}M'_{\mathcal{S}_j} z_{\mathcal{S}_j} = \rho_j z_{\mathcal{S}_j}, & j = 1, 2, \end{cases} \tag{6.30}$$

where $\beta_j, j = 1, 2$ and μ are the Lagrangian multiplier, $\rho_j = \rho_j(\nu_{\mathcal{S}_j}), j = 1, 2$ is the value function and $z_{\mathcal{S}_j} \in \mathbb{R}^{m_j}$ is the positive dominant eigenvector associated to ρ_j. We proceed as follows.

Given that $\mu > 0$, the first condition of (6.30) implies that $\beta \in \mathbb{R}^2$ result strictly positive. Therefore, $\rho_1 = \rho_2 = \xi$.

Henceforth, the proof is analogous to the one of Proposition 4. $\qquad \square$

The most important fact of this proof is that ρ_2 equals ρ_1, that is, the optimal protection levels out the difference in the spectral radius of the two diagonal blocks.

The analysis proceeds in the same way by first solving problem (6.25), where protections on the subset $\mathcal{U}_c \neq \emptyset$ are unconstrained, i.e. $\nu_i > 0, i \in \mathcal{U}_c$, while on the subset $\mathcal{W}_c$ are constrained to the given value d. Notice that if $\mathcal{U}_c = \mathcal{S}_i$, then component $\mathcal{H}_{\mathcal{S}_i}$ could not be protected, and therefore we come back to the case of only one connected component. From the mathematical point of view, if $\rho_i \leq \rho_j, j \neq i$ then the optimal protection will minimize ρ_j until until $\rho_i = \rho_j$.

Result about monotonicity of $\nu^*(c)$ are still valid, and we can come out with an iterative solution similar to (2).

We now consider an application of our result to the case of a network formed by 3 connected components.
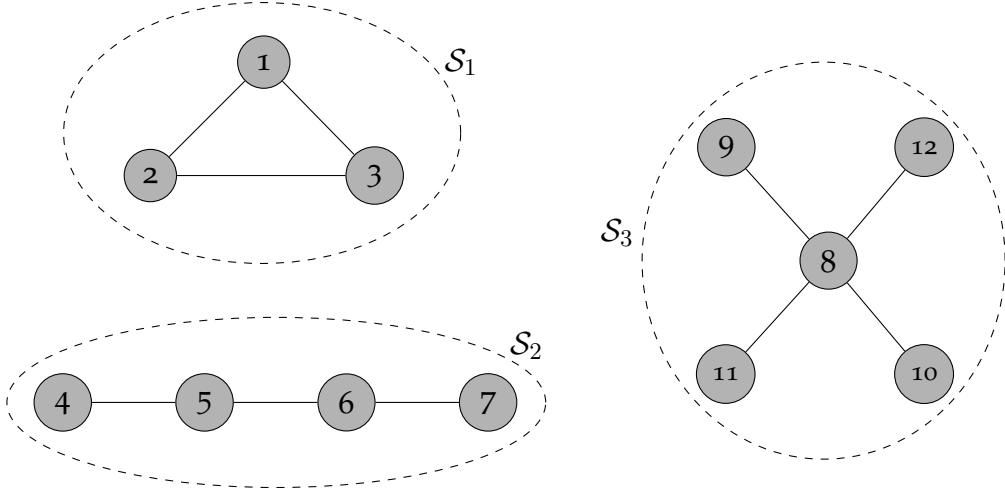
Figure 12: Undirected network with $n = 12$ nodes composed by three connected components.

6.3.3 *Application: absolute displacement*

Consider the network of $n = 12$ agents depicted in Figure 12. $\mathcal{S}_1$ represents the undirected cycle graph of $n = 3$ nodes and we will call regular nodes $1, 2$, and 3. $\mathcal{S}_2$ represents the undirected path graph of $n = 4$ nodes and we will call path$_i$ nodes 5 and 6 and path$_{ex}$ nodes 4 and 7. $\mathcal{S}_3$ represents the undirected star graph of $n = 5$ nodes and we will call hub node 8 and leafs nodes $9, 10, 11$, and 12.

Given the adjacency matrix W, we consider $M = (1 - \lambda)(I - \lambda P)^{-1}$, where P is the normalized weight matrix and $\lambda \in (0, 1)$. As noticed before, matrix M results row-stochastic. During the example we will assume that $\lambda = 0.5$ and $d = \mathbb{1}$.

Considering $\omega \in \mathbb{R}^n$ and let $K = \omega \omega'$. We recall that function Φ is called *absolute displacement* in the context of the Friedkin-Johnsen dynamic. Then, (2) states the exact iterative solution to calculate the optimal protection $\nu^*(c)$ against the worst perturbation K that aims to maximally shift the absolute displacement in the social network depicted in Figure 12.

Figure 13 shows the behavior of the optimal protection $\nu^*(c)$ calculated using the iterative solution of Corollary 2. Notice that $\nu^*(c)$ intervenes in the network without considering the partition of nodes in the three connected components. The most important fact to highlight is that $\nu^*(c)$ starts intervening on the hub of the star graph rather than regular agents of the cycle graph.
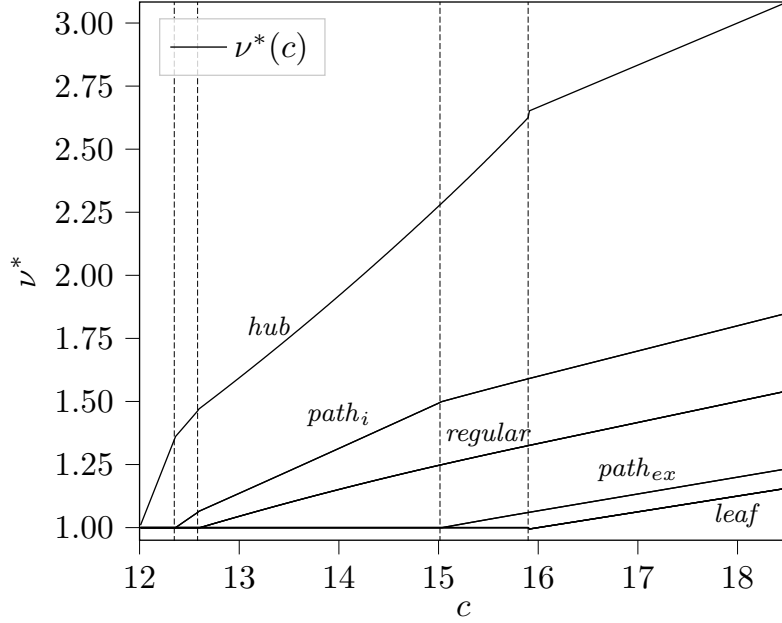
Figure 13: Optimal protection $\nu^*(c)$ on an undirected graph of $n = 12$ composed by three connected components: a cycle graph of $n = 3$ nodes, a path graph of $n = 4$ nodes, and a star graph of $n = 5$ nodes. We have assumed $\lambda = 0.5$ and $d = \mathbb{1}$.

6.4 DISTANCE FROM THE AVERAGE MEASURE

In this section we present and partially solve the optimization problem

$$\min_{\nu \in \mathcal{Q}_c} \max_{K \in \Omega} \, \mathrm{Tr}\left([\nu]^{-1}K[\nu]^{-1}M'(I - J)M\right), \qquad (6.31)$$

that is, problem (6.1) restricted to the case of $Y = I - J$.

Similarly to the previous section, we will denote with H the matrix $M'M$. Also, in this section, we will work under Assumption 2, i.e. that matrix H is irreducible. Given that this part is related to an ongoing problem, this assumption is helpful to ensure that $\mathcal{H}$, the weighted undirected graph associated with matrix H is composed of only one connected component.

We also consider a normalization assumption on matrix M. In particular, we assume that matrix M is row-stochastic. Even if this assumption removes total generality to the model, a lot of applications, see for example Chapter 3, implicitly define M as a row-stochastic matrix. Therefore, this normalization is in-line with applicative contexts.

For the convenience of presentation, it is useful to define the matrix

$$\hat{M} = (I - J)M \,.$$

It then follows that

$$\hat{H} = \hat{M}'\hat{M} \in \mathbb{R}^{m \times m} \tag{6.32}$$

is a positive semi-definite symmetric matrix and is a rank-one update of matrix $M'M$. In fact, notice that we could write $\hat{H}$ as

$$\hat{H} = H - n^{-1}vv' \,,$$

where, in concordance with Section 6.2, we have put $v = M'\mathbb{1}$. Given that M is row-stochastic, it results that the vector $\mathbb{1}$ is the eigenvector associated with eigenvalue 0. In fact,

$$\hat{H}\mathbb{1} = M'(I - J)M\mathbb{1} = M'(I - J)\mathbb{1} = 0\mathbb{1} \,.$$

Assumption 2 also implies that eigenvalue 0 has multiplicity one. In fact, the matrix $\hat{H}$ results in rank $m - 1$, and only vectors of type $\alpha\mathbb{1}, \alpha \in \mathbb{R}$ lay on the kernel of $\hat{H}$.

We recall that $F_{(I-J)}$ coincides with measure Ψ and, in this section, we reuse this symbol to identify the objective function of (6.31). We define the function $\psi : \mathcal{Q}_c \to \mathbb{R}$

$$\psi(\nu) = \max_{K \in \Omega} \Psi(\nu, K), \quad \nu \in \mathcal{Q}_c \,. \tag{6.33}$$

We will denote the spectrum of matrix $A \in \mathbb{R}^{m \times m}$ with $\sigma(A) = \{\sigma_1(A), \sigma_2(A), \ldots, \sigma_m(A)\}$, assuming a non-increasing order in eigenvalues $\sigma_1(A) \geq \cdots \geq \sigma_m(A) \geq 0$. The next result gathers an important fact on the function ψ.

Lemma 6.4.1. *Let $H = M'M, M \in \mathbb{R}^{n \times m}$, M row-stochastic, satisfies Assumption 2. Then, for every ν in $\mathcal{Q}_c$, we have*

(i) $\psi(\nu) = \rho\left([\nu]^{-1}\hat{H}[\nu]^{-1}\right) = \rho\left(\hat{M}\,[\nu]^{-2}\,\hat{M}'\right) \,.$

(ii) *$\psi(\nu)$ is a semi-simple eigenvalue of $[\nu]^{-1}\hat{H}[\nu]^{-1} \,.$*

(iii) *$\psi(\nu)$ is convex in ν.*

(iv) $\sigma_2\left(M[\nu]^{-2}M'\right) \leq \sigma_1\left(\hat{M}[\nu]^{-2}\hat{M}'\right) \,.$

Proof. Appendix 8. □

We now highlight important facts that arise in studying the largest eigenvalue of matrix $\hat{M}[\nu]^{-2}\hat{M}'$. Consider ν in $\mathcal{Q}_c$ and define the matrix function

$$\Pi : \mathcal{Q}_c \to \mathbb{S}^+, \quad \Pi(\nu) = \hat{M}[\nu]^{-2}\hat{M}' \in \mathbb{R}^{n \times n}, \qquad (6.34)$$

and notice that matrix $\Pi(\nu)$ should be rewritten as

$$\Pi(\nu) = \sum_{i=1}^{m} \frac{1}{\nu_i^2} \hat{M}_i \hat{M}_i'$$

that is a combination of symmetric matrices weighted by ν_i^{-2}. Given that matrix Π depends on variable ν, it results that also its spectrum depends on ν. Therefore, eigenvalues of $\Pi(\nu)$ are functions of ν and will be denoted with

$$\sigma(\nu) = \{\sigma_1(\nu), \sigma_2(\nu), \ldots, \sigma_m(\nu)\}, \quad \sigma_i : \mathcal{Q}_c \to \mathbb{R}_+,$$

assuming a non-increasing order

$$\sigma_1(\nu) \geq \sigma_2(\nu) \geq \cdots \geq \sigma_m(\nu).$$

Notice that elements of $\Pi(\nu)$ are analytic in $\nu \in \mathbb{R}^m_{++}$, and therefore its eigenvalues result continuous. However, it is well known [33, 49, 53] that any eigenvalue function σ_i may be non differentiable when it coalesces with others.

An example of this phenomenon is that of matrix $A(x) = [x, -x]$ where $x \in \mathbb{R}$. Then $\sigma_2(x) = -|x|$ and $\sigma_1(x) = |x|$. Both eigenvalues are not differentiable at $x = 0$.

If $\sigma_1(\nu)$ is simple for all $\nu \in \mathcal{Q}_c$ then it results also convex and differentiable. The main problem in applying Perron-Frobenius theory, to show that $\sigma_1(\nu)$ is simple, is that $\Pi(\nu)$ has negative entries due to $I - J$. Therefore, it could happen that $\sigma_1(\nu)$ has multiplicity greater than one for some $\bar{\nu} \in \mathcal{Q}_c$, i.e. $\sigma_1(\bar{\nu}) = \cdots = \sigma_j(\bar{\nu}) > \sigma_{j+1}(\bar{\nu})$ and hence $\psi(\nu)$ results non-differentiable in $\bar{\nu}$.

In the next section, we analyze and specific case of problem

$$\min_{\nu \in \mathcal{Q}_c} \psi(\nu),$$

where we are able to overcome the problem of non-differentiability of $\psi(\nu)$.

We now introduce notation common to next part. Since $\mathcal{Q}_c$ is convex, Lemma 6.4.1 implies that there exists a minimum of ψ. We denote by $\nu^*(c)$ a vector such that

$$\psi(\nu^*(c)) = \min_{\nu \in \mathcal{Q}_c} \psi(\nu)$$

to remember its dependence on the budget c.

Throughout the next sections, we will assume that d is proportional to the vector of all ones, i.e. $d = \underline{d}\mathbb{1}$. These assumptions give us the possibility to better focus on the network side more than the optimization part.

6.4.1 *Special case: polarization in the normalized Friedkin-Johnsen model*

We now consider the special case of doubly-stochastic matrix M. Even if this case seems too specific, we now show an example applied to the Friedkin-Johnsen model that our analysis will comprehend. This setting is used in papers [25, 36, 39, 48], and is sometimes referred to as the normalized Friedkin-Johnsen model.

Example. Consider an undirected graph of n nodes described by the symmetric weight matrix $W \in \mathbb{R}_+^{n \times n}$. Consider Friedkin-Johnsen model (3.11) setting $P = [w]^{-1}W$ (the normalized weight matrix of the graph (2.1)) and $[\lambda] = [\mathbb{1} + w]^{-1}[w]$, where $w = W\mathbb{1}$ is the degree vector. Then, matrix $M = (I - [\lambda]P)^{-1}(I - [\lambda])$ becomes

$$M = (I + L)^{-1},$$

where $L = [w] - W$ is the *Laplacian* matrix of the graph. Given that W represents an undirected graph, it results symmetric. Hence, $(I + L)^{-1}$ is symmetric. Moreover, given that $L\mathbb{1} = 0\mathbb{1}$, matrix M results stochastic and, consequently, doubly-stochastic.

Given that we are at starting stage, we consider the homogeneous setting $d_i = \underline{d}$ for all nodes. This assumption is motivated by the work [36] where authors set the positivity constraint on the intervention, $\nu > 0$. Using reasoning done in Section 4.2, we notice that the case $d = \underline{d}\mathbb{1}$ comprehend also the case analyzed in [36].

The following result is the most important of this section.

Theorem 6.4.2. *Let $H = M'M, M \in \mathbb{R}^{n \times m}$, M doubly-stochastic, satisfies Assumption 2. Assume $d = \underline{d}\mathbb{1}, \underline{d} \in \mathbb{R}_{++}$. Then, for every $c \geq \|d\|^2$ we have*

$$\nu^*(c) = \frac{c}{\sqrt{m}}\mathbb{1}, \quad \psi(\nu^*(c)) = \frac{\sqrt{m}}{c}\sigma_2(H). \tag{6.35}$$

Proof. We will show that if ν is proportional to vector $\mathbb{1}$, then item (iv) of Lemma 6.4.1 holds at the equality.

Given that M is doubly-stochastic, then it results that

$$H\mathbb{1} = \mathbb{1}.$$

Consider $\nu = \mathbb{1}$ and compute as follows:

$$\rho\left([\nu]^{-1}\bar{H}[\nu]^{-1}\right) = \rho\left(H - n^{-1}M'\mathbb{1}\mathbb{1}'M\right)$$
$$= \rho\left(H - J\right).$$

We now prove that $\rho\left(H - J\right)$ is exactly equal to $\sigma_2(H)$. Given that $\mathbb{1}$ is the eigenvector of H associated to the maximum eigenvalue 1, we could use result ([31][Theorem 2.1]). In particular, we could rewrite the spectrum of $\bar{H}$ as

$$\sigma(\bar{H}) = \left\{\sigma_1(H) - n^{-1}\mathbb{1}'\mathbb{1}, \sigma_2(H), \dots, \sigma_m(H)\right\}.$$

Given that H is stochastic we could conclude that

$$\sigma_1(H) - n^{-1}\mathbb{1}'\mathbb{1} = 0,$$

and therefore $\sigma_2(H) = \sigma_1(\bar{H})$.

We now have to rescale the vector $\mathbb{1}$ such that it belongs to $\mathcal{Q}_c$. Consider $\alpha > 0$. We have to set α such that $\alpha^2\mathbb{1}'\mathbb{1} \leq c$ and $\alpha\mathbb{1} \geq \mathbb{1}$. Given that ψ is non-increasing in ν the last inequality results an equality and therefore $\alpha = c/\sqrt{m}$. The assumption on the nature of d ensures that $c/m \geq \underline{d}$. Therefore, if $\nu = c/\sqrt{m}\mathbb{1}$ and M is doubly-stochastic then

$$\sigma_2\left(M[\nu]^{-2}M'\right) = \sigma_1\left(\hat{M}[\nu]^{-2}\hat{M}'\right).$$

$\square$

Remark. Similarly to previous sections, given that M is assumed doubly-stochastic, the optimal solution $\nu^*(c)$ results independent of the network. However, $\psi(\nu^*(c))$ depends on network topology through the second highest eigenvalue of matrix $M'M$.

Differently from $\rho(M'M)$, which is equal to 1 for a doubly-stochastic matrix M, $\sigma_2(M'M)$ differs, in general, among regular networks. Therefore, it is possible to classify networks' structure and study which topology shows lower values of this quantity.

7

CONCLUSIONS AND FUTURE RESEARCH

7.1 CONCLUSION

In this dissertation, we have proposed and discussed a general formulation of an adversarial min-max problem on linear network systems. The two players of the problem, the attacker and the defender, could manipulate the exogenous input vector in order to shift in their desired way a specific performance measure, formulated as a quadratic form of the equilibrium configuration of the system. We have studied different scenarios of the problem, varying the desired performance (e.g. aggregate volatility, absolute displacement, polarization, etc.) and the subspace of admissible attack, formalized as a subsets of the set of symmetric positive semi-definite matrices. Emerging from the analysis are novel network centrality measures that indicate the most influential agents on which the defender should mainly intervene.

In the first part of the thesis, composed of Chapter 3 and Chapter 4, we have formalized the adversarial min-max problem using a general theory that has the credit of embedding many applicative contexts based on linear network models. The main message coming from this part is the formulation of all quadratic performances in an unified way that shows the dependence on the two leading objects of our analysis: the vector of protections ν and the matrix of attack K. On the one hand, our approach can be used to provide general results for those classes of optimization problems and, on the other hand, can be tailored to suit the specificity of the various applications.

In the second part of the thesis, we have adapted our general theory to represent some specific scenarios. In Chapter 5, we have analyzed and solved the adversarial min-max problem by assuming a diagonal matrix K, which studies the specific case of independent perturbations among agents. Within this setting, we have proved that all the performances considered in applications result in a similar form of the external minimization problem and, therefore, could be analyzed using a general formulation. Our main contributions are the followings.

- We have entirely characterized the optimal perturbation $K(\nu)$ of the attacker and the optimal protection $\nu^*(c)$ of the defender. The solutions suggest that the attacker has to focus all the energy on nodes that are most central and

less protected. On the contrary, the defender has to level out diversity in network centralities to make appear agents equally important to the attacker.

- We have exposed sufficient conditions to detect the structure of the optimal protection. The structure of the optimums shows how the optimal protections $\nu^*(c)$ are generally concentrated on a proper subset of nodes. For a fixed budget value c, the optimal protection could be analytically characterized by the water-level function and the centrality vector π that resumes the network topology and depends on the specific performance. In particular, the defender has to exhibit a 'water-filling' structure that levels out the difference created by the network topology. When the lower bound d is equal for all the agents, the subset of nodes on which $\nu^*(c)$ is active is formed by the most important agents that satisfy a condition that depends on the available budget . The optimal intervention becomes much more complicated when the lower bound is heterogeneous among agents.

- We have provided a sensitivity analysis of the optimal protection. The available budget level c generates different regimes in which the subset of agents to be protected changes. More precisely, the defender has to invest its mitigation resources on all nodes if the available budget is sufficiently high or on just a subset of nodes otherwise. Lower is the budget, and less is the number of agents that could be protected. Moreover, the values of c that generate the low and high budget regimes depend on the network topology.

- Centralities that emerged considering the various performance measures are not interchangeable. The most central agent concerning centrality v is not, in general, the most central agent with respect to other centralities ℓ and p. Therefore the subset of agents on which the defender should mainly intervene changes.

In Chapter 6, we have analyzed the adversarial min-max problem assuming only that K belongs to the set of symmetric positive semi-definite matrices. Similarly to the previous chapter, we proved that the external minimization problem should be formulated in a unified form: the spectral radius of a matrix that depends on the specific performance. However, the behavior of the spectral radius considerably changes with respect to the three performance measures Γ, Φ, and Ψ. For this main reason, we have analyzed the three performance separately, obtaining different results. Our main contributions are the followings.

- For the performance γ, we have explicitly characterized the min-max problem's solution.. In particular, we have shown that the spectral radius related

to this measure results in a separable function in ν. This simplification allows studying the external minimization problem using similar tools used in Chapter 5. The centrality measure that naturally emerges from this analysis is the Bonacich centrality, and the optimal intervention of the defender should be derived in an explicit form. We have shown that the optimal interventions $\nu^*(c)$ are generally concentrated on a proper subset of nodes and that observations of Chapter 5 are still valid.

- We completely characterize an explicit recursive solution of the min-max problem related to performance ψ. For this more challenging problem, we have proposed a new iterative solution that exactly solves the problem. For the particular case of the high budget regime, we also gave an exact solution proportional to a new centrality measure. Also, in this setting, the optimal solution we have found results concentrated on a subset of nodes. However, no one of the centralities v, ℓ, or p precisely describes which agents to protect, and, therefore, our result also gives new insights in this direction.

- For the performance ψ, we have entirely characterized the solution of the specific problem of doubly-stochastic matrix M. Even if this setting loses in generality, it is the exact solution to many problems encountered in the literature of Friedkin-Johnsen dynamic models.

To sum up, in this dissertation, we have provided results that completely characterize the structure of the optimal intervention of adversarial min-max problems that consider quadratic performances of the equilibrium state of linear network model that arises in several applications and attracted growing attention in the literature over the past few years. Our results about the structure of the optimal protection also shed new light on how centrality measures naturally arise in targeting problems and how the correlation patterns of the perturbation complicate the intervention.

7.2 CURRENT AND FUTURE RESEARCH

There are several interesting directions in which we could extend and generalize our research.

Primarily, we want complete the analysis of the adversarial min-max problem related to the performance measure Γ. In particular, we would like to extend the obtained result also to not doubly-stochastic matrix M and the general case of heterogeneous lower bounds. We highlight that obtaining a result for this optimization problem could have an impact on two fields: the dynamic opinion field and

the eigenvalues optimization field. As we have already presented, diminishing the polarization in the social context is of increasing importance, and a growing literature is working on it [36]. In the field of eigenvalues optimization, most of the literature is focused on producing practical and general algorithms [16]. However, most of these numerical methods do not give information on the structure of the optimal solutions and do not perform fast when the order of the matrix, i.e. the size of the network, increases. We hope our future results could gain some insight into the structure of the optimal intervention, even in problems where an iterative solution is challenging to obtain.

Another interesting line of research would be undertaking a more formal approach to the adversarial min-max problem. In this dissertation, to maintain a simple approach, we have focused our attention only on linear constraints and an objective function that depends on the protections as $1/\nu$. Much more complicated functions, both on constraints and the objective, could be considered and are already present in the literature [37]. In particular, a possible direction could be to undertake a more formal approach where properties of constraint and objective functions are related to the structure of the solution. In ongoing work, we have obtained general results in the case of separable continuous functions that are convex increasing in the constraint and concave decreasing in the objective. We are still studying the much more general case, i.e. when variables could be paired.

Related to this last part, another interesting problem we are studying is the targeting problem of optimal protections assuming that matrix K is given. This problem could be cast in the setting where perturbations are stochastic vectors and the defender knows the covariance matrix. Within this setting, the objective function becomes a quadratic form where the matrix of interest is the Schur product of a matrix representing the network and the covariance matrix of perturbations. Given that the problem is convex in the protections, many algorithms could be used to find the optimal solution. However, the main issue is finding a relation between network topology and optimal protection. More precisely, we are investigating a measure that could couple together the importance of a node and the action of an external perturbation.

The last research direction we are undertaking is the sensitivity analysis in the parameter d. As we have seen in the examples presented in this dissertation, the heterogeneity in d significantly impacts the structure of the optimal solution. The most crucial case to cite is when the network is regular. We have shown that even in this case, the optimal solution does not act on all the agents equally, and it seems that also the difference in parameters is propagated through the network. Similarly to the analysis we have done for the budget level c, our next step is to analyze this phenomenon by drawing a sensitivity analysis.

APPENDIX

Proof of Lemma 6.3.1

Proof. Consider ν in $\mathcal{Q}_c$.

(i) The first identity follows from Lemma 4.3.1. The second one follows from the property that non-zero eigenvalues of the product of two matrices are invariant with respect the order of the factors.

(ii) It follows from Assumption 2 that $[\nu]^{-1}H[\nu]^{-1}$ is irreducible. Since it is a nonnegative matrix, its dominant eigenvalue is simple [10, Theorem 1.4].

(iii) To prove the third item consider the function $f : \mathbb{R}_+^n \to \mathbb{R}$ given by

$$f(s) := \rho\left(M[s]M'\right). \tag{8.1}$$

Since we can express

$$f(s) = \max_{y \in \mathbb{R}^n : \|y\|_2 = 1} \|M[s]M'y\|_2$$

it follows that $f(s)$, being the max of convex functions, is convex. The same expression yields, since M has all non-negative elements and the maximum is reached by a non-negative vector k, that $f(s)$ is non decreasing with respect to the component-wise order in $\mathbb{R}_+^n$.

We now prove convexity of $\phi(\nu)$. Notice that we can write $\phi(\nu) = f(\nu^{-2})$ where we indicate with ν^{-2} the vector of component-wise inversion of the squares of elements of vector ν. Fix now $\nu_1, \nu_2 \in \mathbb{R}_+^n$ and $\lambda \in [0, 1]$ and notice that, since the inversion function $\nu \mapsto \nu^{-2}$ is convex on $\mathbb{R}_+$,

$$(\lambda\nu_1 + (1-\lambda)\nu_2)^{-2} \leq \lambda\nu_1^{-2} + (1-\lambda)\nu_2^{-2} \tag{8.2}$$

component-wise. We now compute as follows

$$
\begin{aligned}
\phi(\lambda\nu_1 + (1-\lambda)\nu_2) &= f((\lambda\nu_1 + (1-\lambda)\nu_2)^{-2}) \\
&\leq f(\lambda\nu_1^{-2} + (1-\lambda)\nu_2^{-2}) \\
&\leq \lambda f(\nu_1^{-2}) + (1-\lambda)f(\nu_2^{-2}) \\
&= \lambda\phi(\nu_1) + (1-\lambda)\phi(\nu_2),
\end{aligned}
$$

where in the first inequality we have used (8.9) and the monotonicity of f and in the second inequality the convexity of f. Strict convexity follows from the fact

that for every $i = 1, \ldots, n$, there exists h such that $M_{ih} > 0$. Then the function f is strictly increasing and thus the first inequality is strict.

(iv) follows by applying Implicit function theorem and explicit computation could be found in [47]. □

Proof of Proposition 5

Proof. Proof is analogous to the one of Proposition 4. We notice that also in this case, the optimum will lay on the boundary $\|\nu\|^2 = c$. We write $(\nu = \tilde{\nu}, d)$ and we similarly derive first order conditions.

$$
\begin{cases}
-2\tilde{\nu}_i^{-3}(M'z)_i^2 + 2\mu\,\tilde{\nu}_i = 0 & i \in \mathcal{U} \\
\|\tilde{\nu}\|^2 = c - \|d\|^2 \\
M_{\mathcal{U}}[\tilde{\nu}]^{-2}M'_{\mathcal{U}}z + M_{\mathcal{W}}[d]^{-2}M'_{\mathcal{W}}z = \rho z\,.
\end{cases}
\tag{8.3}
$$

The first set of equations yield

$$
\mu^{1/2}[\tilde{\nu}]^2\mathbb{1} = M'_{\mathcal{U}}z\,.
\tag{8.4}
$$

Substituting in the third equation, we obtain

$$
\mu^{1/2}M_{\mathcal{U}}\mathbb{1} + M_{\mathcal{W}}[d]^{-2}M'_{\mathcal{W}}z = \rho z\,.
\tag{8.5}
$$

Notice now that, since $\mathcal{U} \neq \emptyset$, $(M_{\mathcal{W}}[d]^{-2}M'_{\mathcal{W}})_{ij} \leq (M\,[\nu]^{-2}\,M')_{ij}$ for all i and j with strict inequality on the diagonal terms. This implies that $\rho(M_{\mathcal{W}}[d]^{-2}M'_{\mathcal{W}}) < \rho(M\,[\nu]^{-2}\,M') = \rho$ (for instance representing the spectral radius as norm matrix, details can be found in [42, Theorem 8.1.18]. Consequently,

$$
z = \mu^{1/2}(\rho I - M_{\mathcal{W}}[d]^{-2}M'_{\mathcal{W}})^{-1}M_{\mathcal{U}}\mathbb{1}.
\tag{8.6}
$$

Relation (8.6) together with (8.4) yield the thesis. □

Proof of Theorem 6.3.3

Before proving Theorem 6.3.3, we state and proof a simple lemma.

Lemma 8.0.1. *Consider the solution $\nu^{\mathcal{U}}(c)$ of problem (6.25). For every $i \in \mathcal{U}$, $\nu_i^{\mathcal{U}}(c)$ is strictly increasing in c.*

Proof. Put $\nu^{\mathcal{U}}(c) = (\tilde{\nu}, d)$. From the first relation of (6.26), we can write

$$
\tilde{\nu} = \sum_{j=0}^{\infty} \frac{1}{(\phi(\nu^{\mathcal{U}}(c)))^{j+1}}\, M'_{\mathcal{U}}\left(M_W[d]^{-2}M'_{\mathcal{W}}\right)^j M_{\mathcal{U}}\mathbb{1}\,.
$$

Since the value function $\phi(\nu^{\mathcal{U}}(c))$ is strictly decreasing in c, the result is proven. □

We are now ready to prove Theorem 6.3.3.

Proof. We start with continuity. Fix $c \geq \|d\|^2$ and consider a sequence $c_k \to c$ (with $c_k \geq \|d\|^2$) such that $\nu^*(c_k) \to \bar{\nu}$ for $k \to +\infty$. Consider now any sequence $\bar{\nu}_k \in \mathcal{Q}_{c_k}$ such that $\bar{\nu}_k \to \nu^*(c)$ for $k \to +\infty$. Since $\phi(\nu)$ is continuous, we have that $\phi(\nu^*(c_k)) \to \phi(\bar{\nu})$ and $\phi(\bar{\nu}_k) \to \phi(\nu^*(c))$. Since by construction $\phi(\nu^*(c_k)) \leq \phi(\bar{\nu}_k)$ for every k, it holds $\phi(\bar{\nu}) \leq \phi(\nu^*(c))$ and thus $\phi(\bar{\nu}) = \phi(\nu^*(c))$. Since $\phi(\nu)$ is strictly convex, it follows that $\bar{\nu} = \nu^*(c)$. This proves continuity.

We then prove monotonicity. Given any $\bar{c} \geq \|d\|^2$, we now show that $\mathcal{W}_c = \mathcal{S} \backslash \mathcal{U}_c$ is locally constant in a left and, respectively, in a right neighborhood of $\bar{c}$. To this aim, we consider

$$s_{\bar{c}}^- = \liminf_{c \to \bar{c}-} |\mathcal{W}_c| \quad s_{\bar{c}}^+ = \liminf_{c \to \bar{c}+} |\mathcal{W}_c|.$$

Since $s_{\bar{c}}^-$ and $s_{\bar{c}}^+$ are integer-valued, there exists $\delta > 0$ such that

$$
\begin{aligned}
|\mathcal{W}_c| \geq s_{\bar{c}}^- \quad &\forall c \in [\bar{c} - \delta, \bar{c}[\\
|\mathcal{W}_c| \geq s_{\bar{c}}^+ \quad &\forall c \in]\bar{c}, \bar{c} + \delta].
\end{aligned}
\tag{8.7}
$$

Consider any $c_1 \in [\bar{c} - \delta, \bar{c}[$ such that $|\mathcal{W}_{c_1}| = s_{\bar{c}}^-$. Since $\nu^*(c)$ is continuous, there exists $c_2 > c_1$ such that $\mathcal{W}_c \subseteq \mathcal{W}_{c_1}$ for every $c \in [c_1, c_2[$. Suppose we have chosen the supremum of such $c_2 \leq \bar{c}$. By the way c_1 was chosen and the first inequality in (8.7), we actually have that $\mathcal{W}_c = \mathcal{W}_{c_1}$ for $c \in [c_1, c_2[$. Since $\mathcal{W}_c$ is constant on $[c_1, c_2[$, it follows by continuity that $\mathcal{W}_{c_2} \supseteq \mathcal{W}_{c_1}$. Applying monotonicity of Lemma 8.0.1 we thus have $\mathcal{W}_{c_2} = \mathcal{W}_{c_1}$. If $c_2 < \bar{c}$, repeating the same argument, we could further extend the interval $[c_1, c_2[$ on the right where $\mathcal{W}_c$ remains constant in contradiction with the way c_2 was chosen. Therefore $c_2 = \bar{c}$ and we have proven that $\mathcal{W}_c$ is constant on $[c_1, \bar{c}]$. We now consider the right neighborhood. We fix any $c_1 \in]\bar{c}, \bar{c} + \delta[$ such that $|\mathcal{W}_{c_1}| = s_{\bar{c}}^+$ and arguing as above we determine an interval $[c_1, c_2]$ on which $\mathcal{W}_c$ remains constant. By the definition of $s_{\bar{c}}^+$, we can fix c_1 arbitrarily close to $\bar{c}$ and this proves that $\mathcal{W}_c$ is constant on an interval $]\bar{c}, c_2]$. Lemma 8.0.1 guarantees that on the two intervals $[c_1, \bar{c}]$ and $]\bar{c}, c_2]$ the optimal solution $\nu^*(c)$ is non-decreasing. Being continuous, it is non-decreasing in the neighborhood $[c_1, c_2]$ of $\bar{c}$. Finally, being locally non-decreasing and continuous, $\nu^*(c)$ is globally non-decreasing on $[\|d\|^2, +\infty[$. □

Proof of Lemma 6.4.1

Proof. Consider $\nu \in \mathcal{Q}_c$.

(i) The first identity follows from Lemma 4.3.1. The second one follows from the property that non-zero eigenvalues of the product of two matrices are invariant with respect the order of the factors and that matrix $(I - n^{-1}\mathbb{1}\mathbb{1}')$ is symmetric and idempotent.

(ii) It follows from Assumption 2 that $[\nu]^{-1}H[\nu]^{-1}$ is positive definite and hence has full rank m. Given that matrix $[\nu]^{-1}vv'[\nu]^{-1}$ has rank one then

$$
\begin{aligned}
\text{rank}\left([\nu]^{-1}\hat{H}[\nu]^{-1}\right) &\geq \text{rank}\left([\nu]^{-1}H[\nu]^{-1}\right) - \text{rank}\left([\nu]^{-1}vv'[\nu]^{-1}\right) \\
&= m - 1.
\end{aligned}
$$

Noticing that the vector ν belongs to the kernel of $[\nu]^{-1}\hat{H}[\nu]^{-1}$ we can conclude that each eigenvalue of this matrix has geometric multiplicity equal to algebraic multiplicity, and hence is semi-simple.

(iii) To prove the third item consider the function $f : \mathbb{R}_+^m \to \mathbb{R}$ given by

$$
f(s) := \rho\left(\hat{M}[s]\hat{M}'\right). \tag{8.8}
$$

Since we can express

$$
f(s) = \max_{y \in \mathbb{R}^n : \|y\|_2 = 1} \|[s]^{1/2}\hat{M}'y\|_2
$$

it follows that $f(s)$, being the max of convex functions, is convex. Notice that $f(s)$ is non decreasing with respect to the component-wise order in $\mathbb{R}_+^m$. In fact, consider $s' = s + \epsilon$ with $\epsilon_i \geq 0$. Then

$$
\begin{aligned}
f(s') &= f(s + \epsilon) \\
&= \rho\left(\hat{M}[s + \epsilon]\hat{M}'\right) \\
&= \rho\left(\hat{M}([s] + [\epsilon])\hat{M}'\right) \\
&= \rho\left(\hat{M}[s]\hat{M}' + \hat{M}[\epsilon]\hat{M}'\right).
\end{aligned}
$$

Given that the matrix $\hat{M}[\epsilon]\hat{M}'$ is positive semi-definite we could apply ([42][Corollary 4.3.3.]) to ensures that

$$
f(s) = \rho\left(\hat{M}[s]\hat{M}'\right) \leq \rho\left(\hat{M}[s]\hat{M}' + \hat{M}[\epsilon]\hat{M}'\right) = f(s').
$$

We now prove convexity of $\psi(\nu)$. Notice that we can write $\psi(\nu) = f(\nu^{-2})$ where we indicate with ν^{-2} the vector of component-wise inversion of the vector ν. Fix now $\nu_1, \nu_2 \in \mathbb{R}_{++}^m$ and $\lambda \in [0,1]$ and notice that, since the inversion function $\nu \mapsto \nu^{-2}$ is convex on $\mathbb{R}_{++}$,

$$
(\lambda\nu_1 + (1-\lambda)\nu_2)^{-2} \leq \lambda\nu_1^{-2} + (1-\lambda)\nu_2^{-2} \tag{8.9}
$$

component-wise. We now compute as follows

$$
\begin{aligned}
\psi(\lambda\nu_1 + (1-\lambda)\nu_2) &= f((\lambda\nu_1 + (1-\lambda)\nu_2)^{-2}) \\
&\leq f(\lambda\nu_1^{-2} + (1-\lambda)\nu_2^{-2}) \\
&\leq \lambda f(\nu_1^{-2}) + (1-\lambda)f(\nu_2^{-2}) \\
&= \lambda\psi(\nu_1) + (1-\lambda)\psi(\nu_2),
\end{aligned}
$$

where in the first inequality we have used (8.9) and the monotonicity of f and in the second inequality the convexity of f.

(iv) The last properties follows by ([42][Corollary 4.3.9.]). $\qquad\square$

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